Analytical Solution for Luscher's Problems on Two-Layer Consolidation

Paulo Ivo Braga de Queiroz, Delma de Mattos Vidal

Abstract. In this work, the problem of consolidation of two layers is discussed and analytical solutions for the specific case of Luscher's problems are developed. Literature suggests that they are the first analytical solutions for problems that cannot be solved by the Gray-Barber closed form solutions for calculation of equivalent thickness. Luscher's problems are solved using a generalization of the Sturm-Liouville problems for the solution of the partial differential equations. A discussion about the difficulties in generalizing these solutions is also presented. The solution obtained in this work is quantitatively different from that obtained by Luscher via integrated circuits.
Key words: consolidation, analytical solutions, unidirectional flow, heterogeneous media.

1. Introduction

Although the development of new analytical solutions seems to be an outdated subject of study, it may still be justified by the growing use of numerical methods in engineering practice. As newer and more sophisticated numerical programs for geotechnics are developed, the accuracy of their results needs to be assessed somehow. One of the most straightforward procedures to obtain this purpose is to compare numerical results to analytical solutions. In particular, programs that calculate consolidation by coupling flow and effective stresses may be assessed with the help of analytical solutions like the ones presented in this work.

The linear consolidation of two or more layers does not have simple general solutions, despite what might appear at first glance. The illusory simplicity of this problem may be explained by historical reasons.

The formulation of a homogeneous stratum consolidation, attributed to Karl Terzaghi, may be considered as the beginning of modern soil mechanics. Even undergraduate students in civil engineering must know the solution to this problem. The consolidation of a homogeneous soil layer subject to a uniform load is governed by the equation

$$\frac{\partial u_e}{\partial t}(z,t) = c_v \frac{\partial^2 u_e}{\partial z^2}(z,t) \tag{1}$$

where u_e is the excess pore pressure, t is the time elapsed from the start of load application and z is the depth from soil surface. The coefficient of consolidation c_v is given by

$$c_v = \frac{k_w}{\gamma_w m_v} \tag{2}$$

where k_w is the soil permeability to water, γ_w is the unit weight of water and m_v is the coefficient of volume change of the soil. In classic consolidation problems, the following conditions are frequently defined: as initial condition, excess pore pressure is constant ($u_e = u_0$); boundary conditions, excess pore pressure is null at $z = 0$ (top of the consolidating layer) and no flow takes place at $z = H$ (bottom of the consolidating layer). The solution of this boundary value problem may be written in dimensionless terms as

$$\frac{u_e}{u_0} = \sum_{m=0}^{\infty} \frac{2}{M} \sin(MZ) e^{-M^2 T} \tag{3}$$

where the following dimensionless numbers are defined (Lambe & Whitman, 1979):

$$Z = \frac{z}{H}, \quad T = \frac{c_v t}{H^2} \quad \text{and} \quad M = \frac{\pi}{2}(2m+1) \tag{4}$$

The consolidation of two or more layers is governed by the same equation, with the difference that each layer i may have a different coefficient of consolidation (c_{vi}). Moreover, compatibility conditions regarding settlements (ρ), groundwater flow (q) vertical normal stresses (σ_v) and excess pore pressure (u_e) must be imposed at the interface between layers. Even in a two-layer consolidation problem with the same initial and boundary conditions as those presented previously (in the problem of homogeneous stratum), excess pore pressure and effective stress at the interface between layers will change in time. So, two-layer consolidation problems just cannot be divided into two simple problems of single layer consolidation, except in very special cases.

In this work, the problem of consolidation of two layers is discussed and the analytical solutions for Luscher's problems are developed. Although these solutions have already been published (Queiroz, 2002; Queiroz & Vidal, 2002; Queiroz & Vidal, 2003), their complete development and discussion are published herein for the first time. Moreover, recent studies have led to a new form for these solu-

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tions, which are simpler than the previously published ones.

A short survey of works in analytical solutions for consolidation of two layers follows. Some of these works were not carried out in geotechnics, but in other fields of engineering that use the same equation for one-dimensional problems, like heat and mass diffusion. Luscher's problems are then solved using a generalization of the Sturm-Liouville problems, which is the standard technique to solve linear partial differential equations (Arfken & Weber, 2001). A discussion about the difficulties in generalizing these solutions is also presented.

2. Consolidation of Heterogeneous media

The subject of consolidation of heterogeneous media is not covered in most textbooks on Soil Mechanics, despite the fact that it has been investigated for more than fifty years. Gray (1945) and Barber (1945) proposed a simplified technique to solve this problem in an approximate way. To do so, the i^{th} soil layer is taken as reference, and an equivalent thickness H'_j for each other layer can be calculated with the help of the following equation, (Urzua & Christian, 2002),

$$H'_j = H_j \sqrt{\frac{c_{vi}}{c_{vj}}} \tag{5}$$

where H_j is the real thickness of the j_{th} layer, and c_{vi} and c_{vj} are respectively the coefficients of consolidation of the reference layer and of the layer whose equivalent thickness is being calculated. The equivalent thickness of several soil layers is calculated by summing up all the H'_js. The excess pore pressure at any depth may be obtained by using Fourier series, which is calculated over the equivalent thickness of the layers. However, this procedure is based on an implicit hypothesis that the following relationship between permeabilities k_{wi} and coefficients of consolidation of adjacent layers holds,

$$\frac{k_{wi}}{k_{wj}} = \sqrt{\frac{c_{vi}}{c_{vj}}} \tag{6}$$

otherwise, groundwater flow on either side of the interface between adjacent layers will not be equal. Urzua & Christian (2002) clearly show that, depending on the real value of k_{wi}/k_{wj}, appreciable discrepancies between the approximate and exact excess pore pressure may arise.

Domenico & Clark (1964) proposed the use of electric analog circuits to model the consolidation of two layers which have the same thickness and the same compressibility, with permeability for one layer being four times the value of the other. Figure 1 shows a scheme of the problem proposed by the authors, involving the layers' properties relevant to consolidation (k_{wi} and m_{vi}), the depth of the contact between layers, the depth of the bottom of the second

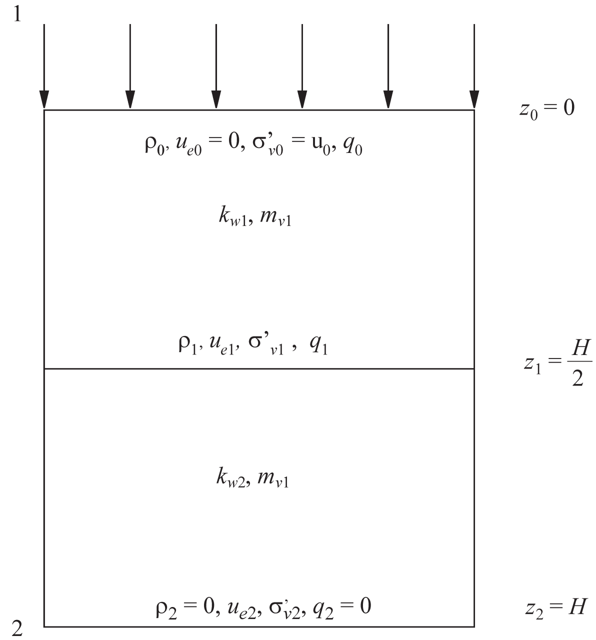

Figure 1 - Scheme of the consolidation of two layers.

layer and the boundary conditions (ρ_i, u_{ei}, σ'_{vi} and q_i). They considered as the initial condition a constant excess pore pressure u_0 and as boundary conditions, a draining frontier at the top (null excess pore pressure) and an impervious frontier at the bottom (no flow). So, two boundary value problems were defined. In the first problem, the upper layer has a higher permeability, while in the second one, the bottom layer is the one with higher permeability. Results obtained by Domenico & Clark were not accurate, which led Luscher (1965) to propose another approach to solve the problem, by using integrated circuits. The results obtained by Luscher were very convincing and his charts became an important reference in the study of consolidation involving multiple layers (Lambe & Whitman, 1979). For this reason, the problems previously described are called Luschers problem in this work, although they were proposed by Domenico & Clark.

Mikhailov & Özisik (1994) proposed a very general method to solve mass and heat diffusion problems in heterogeneous media, for Cartesian, cylindrical and spherical coordinates. Regarding the analogy between one-dimensional consolidation and diffusion in Cartesian coordinates, the methods proposed by these authors can be used to solve consolidation problems involving several layers. Nevertheless, it should be pointed out that this analogy is not valid for problems involving other symmetry conditions, in which consolidation needs to be treated as a tensor problem and its solution may be very different from the solutions of diffusion problems (Mandel, 1953; Abousleiman *et al.*, 1996).

The methods attributed to Mikhailov & Özisik generalize the concepts of eigenvalues and eigenfunctions (from the Sturm-Liouville problems), in order to apply them to problems of diffusion in heterogeneous media. These authors propose that the eigenvalues be obtained by numerical methods such as Newton-Raphson or bisection together with the "sign count method".

3. Eigenvalues for Luscher's problems

3.1. General Formulation

In this section, the eigenvalues of Luscher's problems are obtained in their analytical form, without the use of iterative methods. In order to simplify their equations and the final form of their solution, Luscher's problems are written in their dimensionless forms. So, for each layer j showed in Fig. 1, Eq. (1) becomes

$$\frac{\partial U}{\partial T}=C_j\,\frac{\partial^2 U}{\partial z^2},\ \ j=1,2 \tag{7}$$

where, in analogy to the consolidation problem of a homogeneous medium, the following dimensionless numbers are defined:

$$U=\frac{u_e}{u_0},\ \ Z=\frac{z}{H},\ \ T=\frac{c_{v1}t}{H^2}\ \ \text{and}\ \ C_j=\frac{c_{vj}}{c_{v1}} \tag{8}$$

Here, c_{v1} is the coefficient of consolidation of the upper layer. In order to define boundary and compatibility flow conditions, it will be also useful to put Darcy's law in a dimensionless form,

$$Q=-K_i\,\frac{\partial U}{\partial Z} \tag{9}$$

where the following dimensionless numbers are defined:

$$Q=\frac{\gamma_w H}{k_{w1}u_0}q\ \ \text{and}\ \ K_j=\frac{k_{wj}}{k_{w1}} \tag{10}$$

The eigenvalue problem related to Eq. (7) may be described by equations (Mikhailov & Özisik, 1994)

$$-\mu_i^2\psi_{ij}(Z)=C_j\,\frac{d^2\psi_{ij}}{dZ^2}(Z),\ \ j=1,2 \tag{11}$$

and by the boundary conditions

$$\psi_{i1}(0)=0 \tag{12}$$

$$K_2\,\frac{d\psi_{i2}}{dZ}(1)=0 \tag{13}$$

Here, $\psi_{ij}(Z)$ is the part of the C^1 piecewise continuous eigenfunction, related to the i^{th} eigenvalue μ_i, defined over the depth range of the j^{th} layer. It is also necessary to establish compatibility conditions for flow and excess pore pressure at the interface between layers:

$$Q\left(\frac{1}{2}\right)=-K_1\,\frac{d\psi_{i1}}{dZ}\left(\frac{1}{2}\right)=-K_2\,\frac{d\psi_{i2}}{dZ}\left(\frac{1}{2}\right) \tag{14}$$

$$U\left(\frac{1}{2}\right)=\psi_{i1}\left(\frac{1}{2}\right)=\psi_{i2}\left(\frac{1}{2}\right) \tag{15}$$

The general solution of Eq. (11) is

$$\psi_{ij}=a_{ij}\cos\left(\frac{\mu_i}{\sqrt{C_j}}Z\right)+b_{ij}\sin\left(\frac{\mu_i}{\sqrt{C_j}}Z\right) \tag{16}$$

From the boundary conditions (Eqs. (12) and (13)), this solution becomes:

$$\psi_{i1}=b_i\sin(\mu_i Z) \tag{17}$$

$$\psi_{i2}=a_i\cos\left(\frac{\mu_i}{\sqrt{C_2}}(1-Z)\right) \tag{18}$$

It should be remembered that $K_1 = C_1 = 1$, and which is the reason why these dimensionless numbers do not appear in the previous equations. The eigenvalues μ_i depend on the compatibility conditions (Eqs. (14) and (15)) in the first analysis, and depend on K_2 and C_2 in the second analysis. The following sections deals with the calculation of these eigenvalues.

3.2. Eigenvalues for non-null *Q* and *U* at the interface

In this section, the eigenvalues of Luscher's problems that are related to eigenfunctions that result in Q and U non-null at the interface will be calculated. These eigenfunctions are obtained by applying the compatibility conditions of Q (Eq. (14)) and U (Eq. (15)) on the eigenfunctions obtained from the previous section. As eigenfunctions they are "scalable", that is, they remain eigenfunctions after being multiplied by some real value, one can set $a_i = 1$ by convention.

Once a_i is determined, b_i can be calculated with the help of the compatibility equations for Q at the interface. In this case, by using Eqs. (14), (17) and (18), eigenfunctions become

$$\psi_{i2}=\cos\left(\frac{\mu_i}{\sqrt{C_2}}(1-Z)\right) \tag{19}$$

$$\psi_{i1}=\frac{K_2}{\sqrt{C_2}}\,\frac{\sin\left(\frac{\mu_i}{2\sqrt{C_2}}\right)}{\cos\left(\frac{\mu_i}{2}\right)}\sin(\mu_i Z) \tag{20}$$

These equations will be used later in this paper, in calculations related to Luscher's second problem.

Alternatively, b_i may be calculated with the help of compatibility equations for U at the interface. In this case, by using Eqs. (15), (17) and (18), eigenfunctions become

$$\psi_{i2} = \cos\left(\frac{\mu_i}{\sqrt{C_2}}(1-Z)\right) \tag{21}$$

$$\psi_{i1} = \frac{\cos\left(\frac{\mu_i}{2\sqrt{C_2}}\right)}{\sin\left(\frac{\mu_i}{2}\right)}\sin(\mu_i Z) \tag{22}$$

From the compatibility equation for Q at the interface (Eq. (14)), one has

$$\mu_i \frac{\cos\left(\frac{\mu_i}{2\sqrt{C_2}}\right)}{\sin\left(\frac{\mu_i}{2}\right)}\cos\left(\frac{\mu_i}{2}\right) = K_2 \frac{\mu_i}{\sqrt{C_2}}\sin\left(\frac{\mu_i}{2\sqrt{C_2}}\right) \tag{23}$$

which, after some algebraic operations, becomes

$$\tan\left(\frac{\mu_i}{2\sqrt{C_2}}\right)\tan\left(\frac{\mu_i}{2}\right) = \frac{\sqrt{C_2}}{K_2} \tag{24}$$

In Luscher's first problem, the permeability of the bottom layer is a quarter of the permeability of the upper layer, that is,

$$K_2 = \frac{1}{4} \rightarrow \sqrt{C_2} = \sqrt{\frac{c_{v2}}{c_{v1}}} = \frac{1}{2} \tag{25}$$

By replacing these values in Eq. (24), one obtains

$$\tan(\mu_i)\tan\left(\frac{\mu_i}{2}\right) = \frac{2\tan^2\left(\frac{\mu_i}{2}\right)}{1-\tan^2\left(\frac{\mu_i}{2}\right)} = 2 \tag{26}$$

The previous equation can be solved for $\tan(\mu_i/2)$:

$$\tan\left(\frac{\mu_i}{2}\right) = \pm\frac{\sqrt{2}}{2} \tag{27}$$

It should be pointed out that only positive values of μ_i are of interest, because they will provide a complete set of eigenfunctions of the problem. So, by considering the range ($-\pi/2$, $\pi/2$) as image of the arc tangent function, the following series of eigenvalues are obtained as the solution of Eq. (27):

$$\mu_i = 2\left[\arctan\left(\frac{\sqrt{2}}{2}\right) + (i-1)\pi\right] \quad i = 1,2,3\ldots \tag{28}$$

$$\mu'_i = 2\left[\pi - \arctan\left(\frac{\sqrt{2}}{2}\right) + (i-1)\pi\right] \quad i = 1,2,3\ldots \tag{29}$$

In Luscher's second problem, the permeability of the bottom layer is four times the permeability of the upper layer, that is,

$$K_2 = 4 \rightarrow \sqrt{C_2} = \sqrt{\frac{c_{v2}}{c_{v1}}} = 2 \tag{30}$$

By replacing these values in Eq. (24), one obtains

$$\tan\left(\frac{\mu_i}{2}\right)\tan\left(\frac{\mu_i}{4}\right) = \frac{2\tan^2\left(\frac{\mu_i}{4}\right)}{1-\tan^2\left(\frac{\mu_i}{4}\right)} = \frac{1}{2} \tag{31}$$

The previous equation can be solved for $\tan(\mu_i/4)$:

$$\tan\left(\frac{\mu_i}{4}\right) = \pm\frac{\sqrt{5}}{5} \tag{32}$$

It should be pointed out that only positive values of μ_i are of interest, because they will provide a complete set of eigenfunctions of the problem. So, by considering the range ($-\pi/2$, $\pi/2$) as image of the arc tangent function, the following series of eigenvalues are obtained as the solution of Eq. (32):

$$\mu_i = 4\left[\arctan\left(\frac{\sqrt{5}}{5}\right) + (i-1)\pi\right] \quad i = 1,2,3\ldots \tag{33}$$

$$\mu'_i = 4\left[\pi - \arctan\left(\frac{\sqrt{5}}{5}\right) + (i-1)\pi\right] \quad i = 1,2,3\ldots \tag{34}$$

3.3 On the general solution of the consolidation problem of two layers

It should be noted that the eigenvalues obtained in the previous section are valid only for the specific case that

$$\frac{1-Z_1}{Z_1\sqrt{C_2}} = 2 \quad \text{or} \quad \frac{1-Z_1}{Z_1\sqrt{C_2}} = \frac{1}{2} \tag{35}$$

where Z_1 is the dimensionless depth of the interface between layers (for Luscher's problems, $Z_1 = 1/2$). These particular values permit the use of the formula that calculates the tangent of a double arc, which transforms the compatibility equation into a polynomial in $\tan(\mu_i/2)$ (for Luscher's first problem), or in $\tan(\mu_i/4)$ (for Luscher's second problem). In truth, any rational relation between the factors that multiply μ_i in Eqs. (17) and (18) will produce a polynomial in $\tan(\kappa\mu_i)$, where κ is the greatest common divisor of $(1-Z_1)/\sqrt{C_2}$ and Z_1. If the degree of this polynomial is lower or equal to 4, or if the polynomial belongs to the Galois group of polynomials solvable by radicals (Birkhoff & MacLane, 1977), the problem can readily be solved. There are also methods to calculate the roots of polynomials of fifth and sixth degree using Generalized Hypergeometric and Kampé de Fériet functions (Weisstein, 1999a; Weisstein, 1999b). If $(1-Z_1)/Z_1\sqrt{C_2}$ is not a rational number, μ_i will have to be calculated by the iterative methods proposed by Mikhailov & Özisik (1994).

3.4 Eigenvalues for either *Q* or *U* null at the interface

When $\psi_{ij}(Z)$ causes either Q or U to be null at interface, some algebraic operations performed in section 3.2 might lead to trivial equalities like 0 = 0. These eigenfunctions were discarded by implicit hypotheses in eigenvalue calculations of that section. This section is dedicated to the calculation of eigenvalues related to eigenfunctions of this kind.

Initially, the eigenvalues that result in $U = 0$ at interface are calculated. For Luscher's first problem, this condition is equivalent to (see Eqs. (15), (17) and (18), with $C_2 = 1/4$)

$$\sin\left(\frac{\mu_i}{2}\right)=\cos(\mu_i)=0 \tag{36}$$

This is equivalent to say that $\mu_i/2$ must be an even multiple of $\pi/2$, while μ_i must be an odd multiple of $\pi/2$, which is a contradiction. So, Luscher's first problem does not have eigenvalues of this kind. For Luscher's second problem, the same condition of $U = 0$ at interface is equivalent to (see Eqs. (15), (17) and (18), with $C_2 = 4$)

$$\sin\left(\frac{\mu_i}{2}\right)=\cos\left(\frac{\mu_i}{4}\right)=0 \tag{37}$$

This is equivalent to say that $\mu_i/2$ must be an even multiple of $\pi/2$, while $\mu_i/4$ must be an odd multiple of $\pi/2$. These conditions are obtained by using

$$\mu_i''=4\left[\frac{\pi}{2}+(i-1)\pi\right]\quad i=1,2,3\ldots \tag{38}$$

This is the third series of eigenvalues for Luscher's second problem.

In the following, eigenvalues that result in $Q = 0$ at interface are calculated. For Luscher's first problem, this condition is equivalent to (see Eqs. (14), (17) and (18), with $C_2 = 1/4$)

$$\cos\left(\frac{\mu_i}{2}\right)=\sin(\mu_i)=0 \tag{39}$$

This is also equivalent to say that $\mu_i/2$ must be an odd multiple of $\pi/2$, while μ_i must be an even multiple of $\pi/2$. These conditions are obtained by using

$$\mu_i''=2\left[\frac{\pi}{2}+(i-1)\pi\right]\quad i=1,2,3\ldots \tag{40}$$

This is the third series of eigenvalues for Luscher's first problem. For Luscher's second problem, the same condition of $Q = 0$ at interface is equivalent to (see Eqs. (14), (17) and (18), with $C_2 = 4$)

$$\cos\left(\frac{\mu_i}{2}\right)=\sin\left(\frac{\mu_i}{4}\right)=0 \tag{41}$$

This is equivalent to saying that $\mu_i/2$ must be an odd multiple of $\pi/2$, while $\mu_i/4$ must be an even multiple of $\pi/2$, which is a contradiction. So, Luscher's second problem does not have eigenvalues of this kind.

It should be stressed that the eigenvalues obtained in this section arise only in problems where $(1-Z_1)/Z_1\sqrt{C_2}$ is a rational number, otherwise, no finite value of μ_i will simultaneously satisfy the conditions that both $\mu_i(1-Z_1)/\sqrt{C_2}$ and $\mu_i Z_i$ must be multiples of $\pi/2$.

4. Calculation of the coefficients A_i

After calculating the eigenvalues of Luscher's problems, one can calculate the coefficients $A_i(T)$ that, together with the eigenfunctions ψ_{ij}, will compose the solutions for Luscher's problems that have the following general form:

$$U(Z,T)=\begin{cases}\displaystyle\sum_{i=1}^{\infty}A_i(T)\psi_{i1}(Z)+A_i'(T)\psi_{i1}'(Z)+ \\ \quad A_i''(T)\psi_{i1}''(Z),\quad Z\le\frac{1}{2} \\ \displaystyle\sum_{i=1}^{\infty}A_i(T)\psi_{i2}(Z)+A_i'(T)\psi_{i2}'(Z)+ \\ \quad A_i''(T)\psi_{i2}''(Z),\quad Z>\frac{1}{2}\end{cases} \tag{42}$$

This solution form is slightly different from the original ones proposed by Mikhailov & Özisik (1994), because in this work three different series of eigenvalues were determined, while in the original work, all eigenvalues belong to only one series.

The eigenfunctions of the Sturm-Liouville problem solved in the previous section are orthogonal in relation to the internal product (Tang *et al.*, 1997)

$$(\psi_i,\psi_j)=\int_0^{1/2}\psi_{i1}\psi_{j1}\,dZ+\frac{K_2}{C_2}\int_{1/2}^{1}\psi_{i2}\psi_{j2}\,dZ \tag{43}$$

that is, once the norm is defined (Mikhailov & Özisik, 1994)

$$N_i=(\psi_i,\psi_i)=\int_0^{1/2}\psi_{i1}^2\,dZ+\frac{K_2}{C_2}\int_{1/2}^{1}\psi_{i2}^2\,dZ \tag{44}$$

the following equality holds:

$$\frac{(\psi_i,\psi_j)}{N_i}=\delta_{ij} \tag{45}$$

Here, δ_{ij} is the Kronecker Delta (Arfken & Weber, 2001). Hence, like in the Fourier series, $A_i(0)$ may be calculated by the formula (Mikhailov & Özisik, 1994)

$$A_i(T=0)=\frac{(\psi_i,U_0)}{N_i} \tag{46}$$

where $U_0(Z)$ is the dimensionless initial condition, that in Luscher's problem is equal to 1. So, $A_i(T)$ can be calculated by the formula

$$A_i(T)=A_i(T=0)e^{-\mu_i^2 T} \tag{47}$$

For Luscher's first problem, after some algebraic manipulations, one obtains

$$N_i=\int_0^{1/2}\left[\frac{\cos(\mu_i)}{\sin\left(\frac{\mu_i}{2}\right)}\sin(\mu_i Z)\right]^2 dZ+ \int_{1/2}^{1}\cos^2\left[2\mu_i(1-Z)\right]dZ=\frac{1}{4}+\frac{\cos^2(\mu_i)}{4\sin^2\left(\frac{\mu_i}{2}\right)} \frac{\sin(2\mu_i)}{8\mu_i}\left[3-\frac{1}{\sin^2\left(\frac{\mu_i}{2}\right)}\right] \tag{48}$$

From Eqs. (27) and (40), all values of trigonometric functions in the previous equation can be calculated using double arc trigonometric functions, and the following results may be verified

$$N_i=N'_i=\frac{1}{3} \tag{49}$$

$$N''_i=\frac{1}{2} \tag{50}$$

for any integer i. So, A_i may be calculated as

$$A_i(T=0)=\frac{(\psi_i,U_0)}{N_i}=\frac{(\psi_i,1)}{N_i}= \frac{1}{N_i}\left\{\int_0^{1/2}\frac{\cos(\mu_i)}{\sin\left(\frac{\mu_i}{2}\right)}\sin(\mu_i Z)dZ+\int_{1/2}^{1}\cos[2\mu_i(1-Z)]dZ\right\}= \frac{1}{N_i\mu_i}\left\{\frac{\cos(\mu_i)}{\sin\left(\frac{\mu_i}{2}\right)}\left[1-\cos\left(\frac{\mu_i}{2}\right)\right]+\frac{\sin(\mu_i)}{2}\right\} \tag{51}$$

The absolute value of the trigonometric functions in the previous equation can be calculated from Eqs. (27) and (40). Their sign may be obtained from Eqs. (28), (29) and (40), which provide the quadrants to which the trigonometric operands belong. So, the following results may be verified

$$A_i(T=0)=\frac{1}{N_i\mu_i}\frac{\sqrt{3}}{3}(-1)^{i+1}=\frac{1}{\mu_i}\sqrt{3}(-1)^{i+1} \tag{52}$$

$$A'_i(T=0)=\frac{1}{N'_i\mu'_i}\frac{\sqrt{3}}{3}(-1)^{i+1}=\frac{1}{\mu'_i}\sqrt{3}(-1)^{i+1} \tag{53}$$

$$A''_i(T=0)=\frac{1}{N''_i\mu''_i}(-1)^{i+1}=\frac{1}{\mu''_i}2(-1)^{i+1} \tag{54}$$

For Luscher's second problem, by using ψ_{i1} as defined in Eq. (20), after some algebraic manipulations, one obtains

$$N_i=\int_0^{1/2}\left[2\frac{\sin\left(\frac{\mu_i}{4}\right)}{\cos\left(\frac{\mu_i}{2}\right)}\sin(\mu_i Z)\right]^2 dZ +\int_{1/2}^{1}\cos^2\left[\frac{\mu_i(1-Z)}{2}\right]dZ=\frac{1}{4}+\frac{\sin^2\left(\frac{\mu_i}{4}\right)}{\cos^2\left(\frac{\mu_i}{2}\right)}+ \frac{1}{\mu_i}\left[\sin\left(\frac{\mu_i}{2}\right)-\frac{\sin^2\left(\frac{\mu_i}{4}\right)}{\cos^2\left(\frac{\mu_i}{2}\right)}\sin(\mu_i)\right] \tag{55}$$

The absolute value of the trigonometric functions in the previous equation can be calculated from Eqs. (32) and (38) using double arc trigonometric functions. Their sign may be obtained from Eqs. (33), (34) and (38), which provide the quadrants to which the trigonometric operands belong. So, the following results may be verified

$$N_i=N'_i=\frac{5}{8} \tag{56}$$

$$N''_i=\frac{5}{4} \tag{57}$$

for any integer i. So, A_i may be calculated as

$$A_i(T=0)=\frac{(\psi_i,U_0)}{N_i}=\frac{(\psi_i,1)}{N_i}= \frac{1}{N_i}\left\{\int_0^{1/2}2\frac{\sin\left(\frac{\mu_i}{4}\right)}{\cos\left(\frac{\mu_i}{2}\right)}\sin(\mu_i Z)dZ+\int_{1/2}^{1}\cos\left[\frac{\mu_i(1-Z)}{2}\right]dZ\right\}= \frac{1}{N_i\mu_i}\left\{2\frac{\sin\left(\frac{\mu_i}{4}\right)}{\cos\left(\frac{\mu_i}{2}\right)}\left[1-\cos\left(\frac{\mu_i}{2}\right)\right]+2\sin\left(\frac{\mu_i}{4}\right)\right\} \tag{58}$$

The absolute value of the trigonometric functions in the previous equation can be calculated from Eqs. (32) and (38). Their sign may be obtained from Eqs. (33), (34) and

(38), which provide the quadrants to which the trigonometric operands belong. So, the following results may be verified

$$A_i(T=0)=\frac{1}{N_i\mu_i}\frac{\sqrt{6}}{2}(-1)^{i+1}=\frac{1}{\mu_i}\frac{4\sqrt{6}}{5}(-1)^{i+1} \quad (59)$$

$$A'_i(T=0)=\frac{1}{N'_i\mu'_i}\frac{\sqrt{6}}{2}(-1)^{i+1}=\frac{1}{\mu'_i}\frac{4\sqrt{6}}{5}(-1)^{i+1} \quad (60)$$

$$A''_i(T=0)=\frac{1}{N''_i\mu''_i}2(-1)^{i+1}=\frac{1}{\mu''_i}\frac{8}{5}(-1)^{i+1} \quad (61)$$

5. Solution formulas for Luscher's problems

5.1. Luscher's first problem

Based on the previously calculated values of the eigenvalues, eigenfunctions and series coefficients, the solution of Luscher's first problem is given by:

$$\frac{u_e}{u_0}=\sum_{i=1}^{\infty}\frac{2}{\mu''_i}\sin(\mu''_i Z)e^{-\mu_i''^2 T}+\frac{1}{\mu'_i}\sin(\mu'_i Z)e^{-\mu_i'^2 T} + \frac{1}{\mu_i}\sin(\mu_i Z)e^{-\mu_i^2 T},\ Z\le\frac{1}{2} \quad (62)$$

$$\frac{u_e}{u_0}=\sum_{i=1}^{\infty}(-1)^i\left\{\frac{2}{\mu''_i}\cos[\mu''_i(1-Z)]e^{-\mu_i''^2 T}-\frac{\sqrt{3}}{\mu'_i}\cos[2\mu'_i(1-Z)]e^{-\mu_i'^2 T}-\frac{\sqrt{3}}{\mu_i}\cos[2\mu_i(1-Z)]e^{-\mu_i^2 T}\right\},\ Z>\frac{1}{2} \quad (63)$$

where μ_i, μ'_i and μ''_i are given by Eqs. (40), (29) and (28) and Z and T are defined in Eq. (8).

Figure 2 shows the series solution for eight time instants T. They were calculated with five terms from each series of eigenvalues (i = 1,..., 5), summing up a total of fifteen terms. It may be observed that these curves differ quantitatively from those obtained by Luscher (1965), although the general aspect of the solution is the same. The solution obtained in this work fits very well with results from numerical analyses for C_2 = 1/4 (Queiroz, 2002). Studies under development lead to suppose that Luscher's circuits produced charts valid for C_2 = 1/16, due to a mistake in its project.

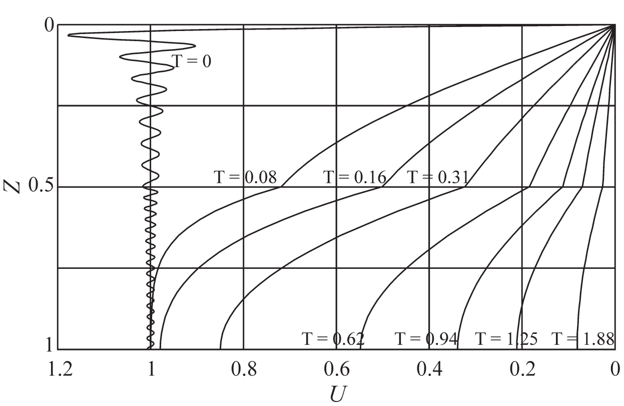

Figure 2 - Series solution for Luscher's first problem.

It should be noted that for $T=0$, the series solution exhibits an oscillation near to origin, with an overshoot of about 18%. This is known as Gibbs Phenomenon (Arfken & Weber, 2001) and it also occurs in the Fourier series, like the one used by Terzaghi in the solution of consolidation of homogeneous media. Calculating U with more terms of the series solution does not decrease the magnitude of overshoot, but only moves it closer to Z = 0. This oscillation only vanishes for high values of T and is not noticeable for T = 0.08.

5.2. Luscher's second problem

Based on the previously calculated values of the eigenvalues, eigenfunctions and series coefficients, the solution of Lucher's second problem is expressed by:

$$\frac{u_e}{u_0}=\sum_{i=1}^{\infty}\frac{16}{5\mu''_i}\sin(\mu''_i Z)+\frac{12}{5\mu'_i}\sin(\mu'_i Z)e^{-\mu_i'^2 T}+\frac{12}{5\mu_i}\sin(\mu_i Z)e^{-\mu_i^2 T},\ Z\le\frac{1}{2} \quad (64)$$

$$\frac{u_e}{u_0}=\sum_{i=1}^{\infty}(-1)^i\left\{\frac{8}{5\mu''_i}\cos\left[\frac{1}{2}\mu''_i(1-Z)\right]e^{-\mu_i''^2 T}-\frac{4\sqrt{6}}{5\mu'_i}\cos\left[\frac{1}{2}\mu'_i(1-Z)\right]e^{-\mu_i'^2 T}-\frac{4\sqrt{6}}{5\mu_i}\cos\left[\frac{1}{2}\mu_i(1-Z)\right]e^{-\mu_i^2 T}\right\},\ Z>\frac{1}{2} \quad (65)$$

where μ_i, μ'_i and μ''_i aregiven by Eqs. (38), (34) and (33) and Z and T are defined in Eq. (8).

Figure 3 shows the series solution for eight time instants T. They were calculated with five terms from each series of eigenvalues. It may be observed that these curves differ quantitatively from those obtained by Luscher (1965), although the general aspect of the solution is the same. The solution obtained in this work fits very well with results from numerical analyses for C_2 = 4 (*e.g.*, Queiroz, 2002). Studies under development lead to suppose that Luscher's circuits produced charts valid for C_2 = 16, due to a mistake in its project.

It should be noted that, as in the solution of first Luscher's problem, Gibbs Phenomenon occurs for T = 0.

6. Final Remarks

In this work, the problem of consolidation of two layers was discussed, and some techniques for its solution were commented. Analytical solutions for two special

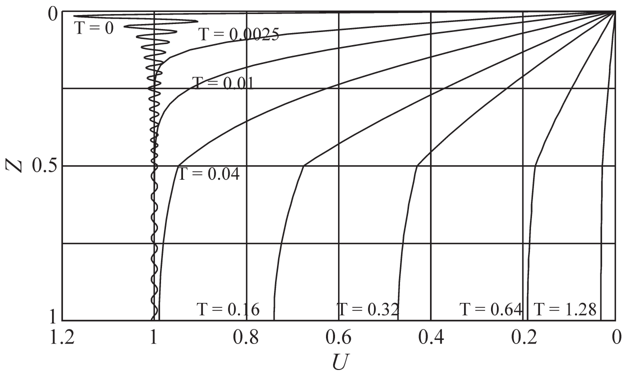

Figure 3 - Series solution for Luscher's second problem.

problems, the so-called Luscher's problems, were developed. These analytical solutions may be useful as benchmark for convergence analyses of numerical methods. As an example, the analytical solution of the first problem furnishes $U = 0.08$ for $T = 1.88$ and $Z = 1$. Luscher (1965) obtained via integrated circuits $U = 0.16$ for the same T and Z, which leads to an error of about 100% over the analytical solution. Future works will provide solutions for other problems in consolidation of heterogeneous media.

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List of Symbols

A_i, A_i' and A_i'': Coefficients of trigonometric series for i^{th} eigenfunction

c_v, c_{vi} and c_{vj}: Coefficients of consolidation (related to i^{th} or j^{th} soil layer)

C_i: Dimensionless coefficient of consolidation of i^{th} soil layer

H, H_j: Thickness of soil layer (thickness of i^{th} soil layer)

H_j': Equivalent thickness of j^{th} soil layer

k_w, k_{wi}, k_{wj}: Soil permeability related to water (permeability of i^{th} or j^{th} soil layer)

K_i: Dimensionless permeability of i^{th} soil layer

M and m: Multiplying factors of Fourier series

m_v: Coefficient of volume change

N_i, N_i' and N_i'': Norm of i^{th} eigenfunction

q: Groundwater flow

Q: Dimensionless groundwater flow

t: Time

T: Dimensionless time

u_0: Initial excess of pore pressure

u_e: Excess of pore pressure

U: Dimensionless excess of pore pressure
z: Depth
Z and Z_i: Dimensionless depth (of the bottom of i^{th} soil layer)
γ_w: Unit weight of water
μ_i: Eigenvalue related to i^{th} eigenfunction
ρ: Settlement
σ_v: Vertical tension
ψ_{ij}: C^1 Piecewise continuous eigenfunction, related to the i^{th} eigenvalue μ_i, defined over the depth range of the j^{th} layer.
$(\bullet,\bullet)$: Internal product of functions