Some Applications of Linear Viscoelasticity to Problems of Consolidation under Variable Loading

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Abstract. The main purpose of this paper is to present a general method to derive closed form solutions for one-dimensional consolidation problems under time dependent loading using the Linear Viscoelasticity theory. A review of the basic concepts of this theory is initially presented and, mainly for illustration purposes, the method is applied to three consolidation problems, leading to relevant solutions for Geotechnical Engineering. In the first application, considering Terzaghi's and Barron's solutions, creep functions are determined for vertical and radial drainage, allowing derivation of expressions for one-dimensional consolidation under a number of linear loads for these drainage conditions. Using Carrillo's equation, the creep function for combined vertical and radial drainage is obtained, leading to corresponding solution for linear variable loading. Partial submersion of embankments on soft soils is another consolidation problem under time dependent loading solved by means of Viscoelasticity. Classical approximate solutions are used in this second application to establish creep functions for vertical drainage and exact Barron's solution to establish creep function for radial drainage. An expression for late stages of consolidation is also derived for combined vertical and radial drainage condition. The third application considers the problem of load transfer from a consolidating deposit of soft clay to a pattern of drain columns of finite stiffness. Diagrams concerning a case of consolidation under linear variable three-step loading and consolidation with partial submersion of the fill are provided to illustrate the solutions obtained.

Key words: one-dimensional consolidation, viscoelasticity, variable loading, submersion of embankments, drain columns.

1. Introduction

The main purpose of this paper is to present a general and simple method to derive solutions for consolidation problems involving variable loading. The method consists in employing Linear Viscoelasticity to state constitutive equations and Laplace transforms to solve these equations. The corresponding creep function for each problem is obtained from existing solutions for constant loading.

Attention should be drawn to the words *viscoelasticity and creep function*, which might be misleading once there is no viscous phenomenon affecting primary consolidation. However, for total stress analysis, the method is perfectly applicable despite the hydrodynamic feature of the process once there is a time dependent strain for a constant total stress applied.

The method is illustrated through three applications involving practical problems of geotechnical engineering design, as explained below.

Construction of embankments on soft soils is usually performed in steps of loading. Owing to conditions such as construction timing and low strength of the underlying soil, these steps must be carefully planned as far as the rate of filling is concerned. Also, in order to abbreviate the time of construction, drain wells are often installed and therefore the analysis must contemplate consolidation with both vertical and radial drainage.

Conventional analyses of consolidation with variable loading usually lead to rather complex mathematical formulations. Nevertheless, a great deal of work has been accomplished in order to obtain solutions for one-dimensional consolidation with variable loading. Therefore, a number of approximate and exact solutions are now available (Terzaghi and Frölich, 1936; Taylor, 1942; Schiffman, 1958; Olson, 1977; Kurma Rao and Vijaya Rama Raju, 1990; Da Mota, 1996; Lekha *et al.*, 1998).

Using total stress linear viscoelastic approach, solutions to this first application may be obtained in an elegant and simple way, even for complex variable loading history and either for vertical, radial or combined drainage conditions. These solutions guarantee unquestionable benefits in accuracy when embankments construction conditions impose several steps of loading with different loading rates.

The second application regards the effect of the submersion of the fill, which is another important problem concerning one-dimensional consolidation with variable loading. Total stress viscoelastic analysis provides a closed form solution for vertical, radial or combined drainage conditions.

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The third application considers the problem of load transfer from a consolidating deposit of soft clay to a pattern of drain columns of finite stiffness.

The analyses presented in this paper consider linear behaviour of soils for small deformation problems. This is certainly an approximation and may introduce some inaccuracy in the results. However, this approximation is the same as that found in the Terzaghi's Theory of Consolidation, which produces reasonably accurate predictions for most of the practical situations. It is worth mentioning that despite Linear Viscoelasticity not applying to materials exhibiting nonlinear stress-strain behaviour, an extension to the linear superposition principle can be employed to derive nonlinear viscoelastic constitutive equations (Findley *et al.*, 1976).

2. Basic Concepts

2.1. Creep and relaxation functions

Many stress-strain-time relations existing in technical literature are basically empirical (Findley *et al.*, 1976; Bland, 1960). Most of them were established to fit experimental data obtained under constant stress and temperature. However, actual behaviour of materials has shown that the strain corresponding to a particular time depends on all the stress values to which the material has been submitted in the past and not on its final value. Therefore, the creep phenomenon is affected by the whole stress history. Considering this, several methods have been proposed to represent the viscoelastic behaviour of the materials. In general, however, there are two alternative mathematical procedures to represent the stress-strain-time behaviour of the materials: differential and integral forms.

The study carried out in this paper employed the integral form or, as frequently referred, hereditary integrals. The advantage of the hereditary integrals over the differential form consists of a higher flexibility of representation of the material properties inferred from laboratorial tests. Integral form can also be extended to describe the behaviour of aging materials and incorporate temperature effects. Besides, for problems involving rather complex time loading functions, the integral method leads to a simpler solution.

In a uniaxial creep test a step of constant stress $\sigma = \sigma_0 H(t)$, where $H(t)$ represents the unit step or Heaviside function, is applied to a viscoelastic material and the strain $\varepsilon(t)$ is measured. For materials exhibiting linear behaviour, the strain can be represented by

$$\varepsilon(t) = \sigma_0 J(t) \tag{1}$$

or

$$J(t) = \frac{\varepsilon(t)}{\sigma_0} \tag{2}$$

The function $J(t)$ is called creep function or creep compliance and is a material property.

In a relaxation test a step of constant strain $\varepsilon = \varepsilon_0 H(t)$ is prescribed to a viscoelastic material and the stress $\sigma(t)$ is measured. For linear materials the stress can be represented by

$$\sigma(t) = \varepsilon_0 R(t) \tag{3}$$

or

$$R(t) = \frac{\sigma(t)}{\varepsilon_0} \tag{4}$$

The function $R(t)$ is called relaxation function or relaxation modulus and, likewise the creep function, is a material property.

2.2. Integral representation of creep for uniaxial stress

If a viscoelastic body with linear behaviour is subjected to a continuous stress function $\sigma(t)$ with finite derivative within the concerned time interval, representing the stress history, the corresponding strain function $\varepsilon(t)$ can be obtained from the equation

$$\varepsilon(t) = \int_0^t J(t-\tau) \frac{\partial \sigma(\tau)}{\partial \tau} d\tau \tag{5}$$

where τ is an auxiliary variable and $J(t - \tau)$ the creep function. Expression (5) may also be alternatively represented by

$$\varepsilon(t) = \sigma(t) J(0) - \int_0^t \frac{\partial J(t-\tau)}{\partial \tau} \sigma(\tau) d\tau \tag{6}$$

if $J(t - \tau)$ is continuous and differentiable.

Expressions (5) and (6) apply to the particular case where the process begins at time $t = 0$ and the initial value of the stress is zero, *i.e.*, $\sigma(0) = 0$. For the general case, with the process beginning at time τ_0 and the initial value of the stress being different from zero, the following equations hold

$$\varepsilon(t) = \sigma(\tau_0) J(t-\tau_0) + \int_{\tau_0}^t J(t-\tau) \frac{\partial \sigma(\tau)}{\partial \tau} d\tau \tag{7}$$

$$\varepsilon(t) = \sigma(t) J(0) - \int_{\tau_0}^t \frac{\partial J(t-\tau)}{\partial \tau} \sigma(\tau) d\tau \tag{8}$$

Once the creep function $J(t)$ is identified, one of the Eqs. (5), (6), (7) and (8) can be employed to predict the stress function $\sigma(t)$ given a prescribed strain history $\varepsilon(t)$. However, resolving $\sigma(t)$ using one of the above equations involves the solution of an integral equation, which is mathematically much more complicated than a direct integration. Otherwise, the following equations can be written

$$\sigma(t)=\varepsilon(\tau_0)R(t-\tau_0)+\int_{\tau_0}^{t}R(t-\tau)\frac{\partial\varepsilon(\tau)}{\partial\tau}\,d\tau \tag{9}$$

$$\sigma(t)=\varepsilon(t)R(0)-\int_{\tau_0}^{t}\frac{\partial R(t-\tau)}{\partial\tau}\varepsilon(\tau)d\tau \tag{10}$$

Therefore, to determine $\sigma(t)$ from a prescribed strain history $\varepsilon(t)$, the relaxation function $R(t)$ must be known.

2.3. The method proposed

The proposed method consists of three steps:

(1) Using Eq. (2) and the relevant consolidation solution for *constant total stress*, define the creep function;

(2) Taking into account the creep function defined in (1) and the total stress history of the problem, the Laplace transform one of the Eqs. (5), (6), (7) and (8) leads to the solution or to an algebraic equation that can be easily solved;

(3) Find the inverse Laplace transform of the solution obtained in (2).

The last step is often the most difficult one, presenting in some cases no closed form solution.

3. First Application: Analysis of One-Dimensional Consolidation under a Number of Linear Variable Loads

The analysis presented in this paper establishes creep functions derived from Terzaghi's and equal strain Barron's equations for vertical and radial drainage respectively. Therefore, the same assumptions made to obtain those equations are also considered here, as follows:

(1) The soil is considered homogeneous and fully saturated;

(2) The compressibility of the soil particles and the water are negligible in comparison with that of the soil structure;

(3) There is only vertical displacement of the soil particles and vertical water flow for Terzaghi's equation and radial water flow for Barron's equation;

(4) Darcy's law is strictly valid;

(5) The stress-strain relationship of the soil structure is linear.

3.1. First case: vertical drainage

Suppose a homogeneous deposit of soft clay (Fig. 1) subjected to a loading constituted by a set of linear variable loads (Fig. 2). According to Terzaghi and Frölich (1936), the average degree of consolidation U_v for one-dimensional consolidation with vertical drainage and constant loading is

$$U_v=1-\sum_{0}^{\infty}\frac{2}{M^2}e^{-M^2T_v} \tag{11}$$

where

$$M=(2m+1)\frac{\pi}{2} \tag{12}$$

$$T_v=\frac{c_v t}{H^2} \tag{13}$$

where c_v = coefficient of consolidation for vertical flow, t = time and H = maximum drainage distance.

It may also be written

$$U_v=\frac{s(t)}{s(\infty)}=\frac{\bar{\varepsilon}(t)}{\bar{\varepsilon}(\infty)} \tag{14}$$

where $s(t)$ = settlement of the top of the layer at time t, $s(\infty)$ = final settlement of the top of the layer, at infinite time, $\bar{\varepsilon}(t)$ = average vertical strain at time t and $\bar{\varepsilon}(\infty)$ = final average vertical strain, at infinite time.

For a homogeneous deposit of soft soil, the average strain at infinite time may be written as

$$\bar{\varepsilon}(\infty)=m_v\sigma_0 \tag{15}$$

where m_v = coefficient of volume compressibility and σ_0 = total stress applied.

Therefore

$$\bar{\varepsilon}(t)=m_v\sigma_0\left(1-\sum_{0}^{\infty}\frac{2}{M^2}e^{-M^2\frac{c_v t}{H^2}}\right) \tag{16}$$

Taking into account expression (16) and recalling that

$$J(t)=\frac{\varepsilon(t)}{\sigma_0} \tag{2}$$

follows

$$\bar{J}_v(t)=m_v\left(1-\sum_{0}^{\infty}\frac{2}{M^2}e^{-M^2\frac{c_v t}{H^2}}\right) \tag{17}$$

Let $q(t)$ be the loading acting on the surface of the clay deposit, as illustrated in Fig. 2. For $t \le t_n$, this loading may be represented by

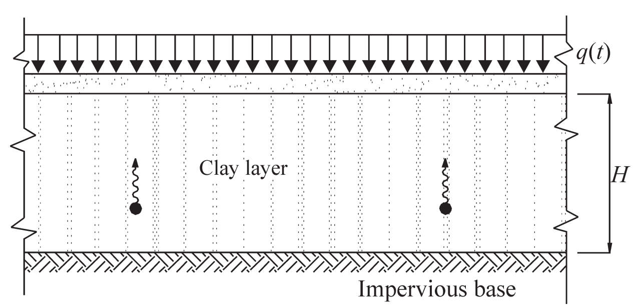

Figure 1 - Representation of a clay layer subjected to one-dimensional consolidation with vertical drainage under variable loading.

$$q(t) = \sum_{1}^{n} \{q_i(t - t_{i-1})[H(t - t_{i-1}) - H(t - t_i)]\} \tag{18}$$

where $t_0 = 0$ and $H(t)$ is the Heaviside function, as previously mentioned.

Equation (6) may be written as

$$\bar{\varepsilon}(t) = q(t)\bar{J}_v(0) - \int_0^t q(\tau)\frac{\partial \bar{J}_v(t-\tau)}{\partial \tau}d\tau \tag{19}$$

or

$$\bar{\varepsilon}(t) = q(t)\bar{J}_v(0) - \sum_{1}^{n}\left\{\int_0^t q_i(\tau - t_{i-1})[H(\tau - t_{i-1}) - H(\tau - t_i)]\frac{\partial \bar{J}_v(t-\tau)}{\partial \tau}d\tau\right\} \tag{20}$$

Considering Eq. (17) and bearing in mind that

$$\bar{J}_v(0) = m_v\left(1 - \sum_{0}^{\infty}\frac{2}{M^2}\right) = 0 \tag{21}$$

$$q_k(\tau - t_{k-1}) = \sum_{1}^{k-1}\Delta q_i + \frac{\Delta q_k}{t_k - t_{k-1}}(\tau - t_{k-1}) \tag{22}$$

the integral of Eq. (20) may be evaluated, giving

$$\bar{\varepsilon}(T_v) = m_v\sum_{1}^{k-1}\left\{\Delta q_i\left[1 - \frac{2}{T_{v_i} - T_{v_{(i-1)}}} \times \sum_{0}^{\infty}\frac{1}{M^4}(1 - e^{-M^2(T_{v_i} - T_{v_{(i-1)}})}) \times e^{-M^2(T_v - T_{v_i})}\right]\right\} + m_v\Delta q_k \times \left[\frac{T_v - T_{v_{(k-1)}}}{T_{v_k} - T_{v_{(k-1)}}} - \frac{2}{T_{v_k} - T_{v_{(k-1)}}}\sum_{0}^{\infty}\frac{1}{M^4}\left(1 - e^{-M^2(T_v - T_{v_{(k-1)}})}\right)\right] \tag{23}$$

and

$$U_v(T_v) = \sum_{1}^{k-1}\left\{\rho_i\left[1 - \frac{2}{T_{v_i} - T_{v_{(i-1)}}}\sum_{0}^{\infty}\frac{1}{M^4} \times \left(1 - e^{-M^2(T_{v_i} - T_{v_{(i-1)}})}\right) \times e^{-M^2(T_v - T_{v_i})}\right]\right\} + \rho_k\left[\frac{T_v - T_{v_{(k-1)}}}{T_{v_k} - T_{v_{(k-1)}}} - \frac{2}{T_{v_k} - T_{v_{(k-1)}}}\sum_{0}^{\infty}\frac{1}{M^4}\left(1 - e^{-M^2(T_v - T_{v_{(k-1)}})}\right)\right] \tag{24}$$

where

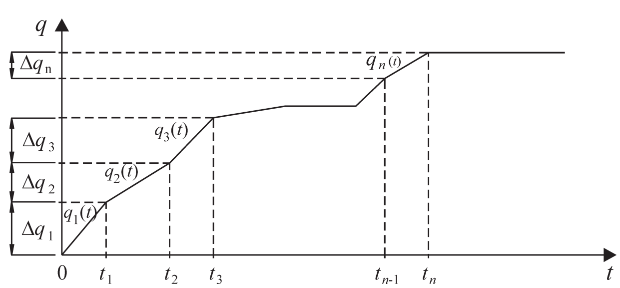

Figure 2 - General loading history considered in the analysis.

$$\rho_i = \frac{\Delta q_i}{\sum_{i}^{n}\Delta q_i} \tag{25}$$

T_{v_i} = time factor for vertical drainage relative to time t_i.

Expressions (23) and (24) are valid for $t_{k-1} \le t \le t_k$ ($k \le n$). For $t > t_n$ the loading history is represented by

$$q(t) = \sum_{1}^{n}\left\{q_i(t - t_{i-1})\left[H(t - t_{i-1}) - H(t - t_i)\right]\right\} + q_n(t_n)H(t - t_n) \tag{26}$$

where

$$q_n(t_n) = \sum_{1}^{n}\Delta q_i \tag{27}$$

Substituting Eq. (26) into Eq. (19) and integrating, yields

$$\bar{\varepsilon}(T_v) = m_v\sum_{1}^{n}\left\{\Delta q_i\left[1 - \frac{2}{T_{v_i} - T_{v_{(i-1)}}} \times \sum_{0}^{\infty}\frac{1}{M^4}(1 - e^{-M^2(T_{v_i} - T_{v_{(i-1)}})}) \times e^{-M^2(T_v - T_{v_i})}\right]\right\} \tag{28}$$

and

$$U_v(T_v) = \sum_{1}^{n}\left\{\rho_i\left[1 - \frac{2}{T_{v_i} - T_{v_{(i-1)}}}\sum_{0}^{\infty}\frac{1}{M^4} \times \left(1 - e^{-M^2(T_{v_i} - T_{v_{(i-1)}})}\right) \times e^{-M^2(T_v - T_{v_i})}\right]\right\} \tag{29}$$

Expressions (28) and (29) are therefore valid for $t \ge t_n$.

3.2. Second case: radial drainage

This case considers a homogeneous deposit of soft clay with vertical drains (Fig. 3) subjected to the same general loading scheme admitted in the first case (Fig. 2). It is assumed that no vertical drainage occurs in the clay.

The degree of consolidation U_r for one-dimensional consolidation with radial drainage, constant loading and equal strain (Barron, 1948) is

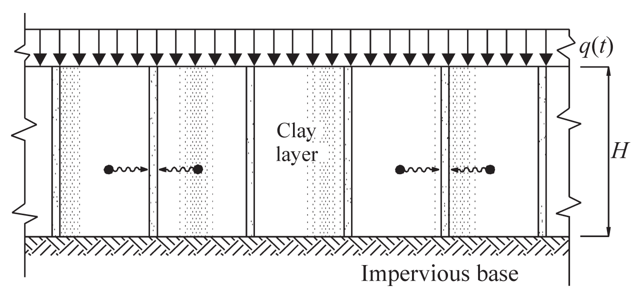

Figure 3 - Representation of a clay layer subjected to one-dimensional consolidation with radial drainage under variable loading.

$$U_r = 1 - e^{-\frac{8T_r}{f(n)}} \tag{30}$$

where

$$T_r = \frac{c_r t}{4r_e^2} \tag{31}$$

$$f(n) = \left[\frac{n^2}{n^2-1}\ln(n) - \frac{3n^2-1}{4n^2}\right] \tag{32}$$

$$n = \frac{r_e}{r_d} \tag{33}$$

where c_r = coefficient of consolidation for radial flow, t = time, r_e = radius of the zone of influence of the drain and r_d = radius (or equivalent radius) of the drain.

Taking into account that $\varepsilon(z, t)$ in this case is constant along the depth z, the following equation may be written

$$U_r = \frac{s(t)}{s(\infty)} = \frac{\varepsilon(t)}{\varepsilon(\infty)} \tag{34}$$

and so

$$\varepsilon(t) = m_v \sigma_0 \left(1 - e^{-\frac{2c_r t}{f(n) r_e^2}}\right) \tag{35}$$

Therefore, the creep function for radial drainage is

$$J_r(t) = m_v \left(1 - e^{-\frac{2c_r t}{f(n) r_e^2}}\right) \tag{36}$$

In this case, Eq. (6) may be written as

$$\varepsilon(t) = q(t) J_r(0) - \int_0^t q(\tau) \frac{\partial J_r(t-\tau)}{\partial \tau} d\tau \tag{37}$$

For $t \le t_n$, the vertical strain $\varepsilon(t)$ due to the loading history $q(t)$ represented by Eq. (18) is

$$\varepsilon(t) = q(t) J_r(0) - \sum_1^n \left\{ \int_0^t q_i(\tau - t_{i-1})[H(\tau - t_{i-1}) - H(\tau - t_i)] \frac{\partial J_r(t-\tau)}{\partial \tau} d\tau \right\} \tag{38}$$

Inserting Eq. (36) into the right-hand side of Eq. (38), recalling that

$$J_r(0) = 0 \tag{39}$$

$$q_k(\tau - t_{k-1}) = \sum_1^{k-1} \Delta q_i + \frac{\Delta q_k}{t_k - t_{k-1}}(\tau - t_{k-1}) \tag{22}$$

and evaluating Eq. (38), yields

$$\varepsilon(T_r) = m_v \sum_1^{k-1} \left\{ \Delta q_i \left[1 - \frac{f(n)}{8(T_{r_i} - T_{r(i-1)})} \left(1 - e^{-\frac{8}{f(n)}(T_{r_i} - T_{r(i-1)})}\right) \times e^{-\frac{8}{f(n)}(T_r - T_{r_i})} \right] \right\} + m_v \Delta q_k \left[\frac{T_r - T_{r(k-1)}}{T_{r_k} - T_{r(k-1)}} - \frac{f(n)}{8(T_{r_k} - T_{r(k-1)})} \times \left(1 - e^{-\frac{8}{f(n)}(T_r - T_{r(k-1)})}\right) \right] \tag{40}$$

and

$$U_r(T_r) = \sum_1^{k-1} \rho_i \left[1 - \frac{f(n)}{8(T_{r_i} - T_{r(i-1)})} \left(1 - e^{-\frac{8}{f(n)}(T_{r_i} - T_{r(i-1)})}\right) \times e^{-\frac{8}{f(n)}(T_r - T_{r_i})} \right] + \rho_k \left[\frac{T_r - T_{r(k-1)}}{T_{r_k} - T_{r(k-1)}} - \frac{f(n)}{8(T_{r_k} - T_{r(k-1)})} \times \left(1 - e^{-\frac{8}{f(n)}(T_r - T_{r(k-1)})}\right) \right] \tag{41}$$

where

$$\rho_i = \frac{\Delta q_i}{\sum_i^n \Delta q_i} \tag{25}$$

T_{r_i} = time factor for radial drainage relative to time t_i.

Expressions (40) and (41) hold for $t_{k-1} \le t \le t_k$ $(k \le n)$. For $t > t_n$, using the loading history represented by Eq. (26), $\varepsilon(t)$ may be expressed by

$$\varepsilon(T_r) = m_v \sum_1^{n} \left\{ \Delta q_i \left[1 - \frac{f(n)}{8(T_{r_i} - T_{r(i-1)})} \left(1 - e^{-\frac{8}{f(n)}(T_{r_i} - T_{r(i-1)})}\right) \times e^{-\frac{8}{f(n)}(T_r - T_{r_i})} \right] \right\} \tag{42}$$

and

$$U_r(T_r) = \sum_1^{n} \rho_i \left[1 - \frac{f(n)}{8(T_{r_i} - T_{r(i-1)})} \left(1 - e^{-\frac{8}{f(n)}(T_{r_i} - T_{r(i-1)})}\right) \times e^{-\frac{8}{f(n)}(T_r - T_{r_i})} \right] \tag{43}$$

Expressions (42) and (43) must be applied when $t > t_n$.

3.3. Third case: combined vertical and radial drainage

For combined vertical and radial drainage and constant loading Carrillo (1942) has proved that

$$(1 - U_{vr}) = (1 - U_v) \times (1 - U_r)^{1} \tag{44}$$

1 For the sake of simplicity the representation U and J is employed instead of $U(t)$ and $J(t)$.

where U_{vr} = average degree of consolidation for combined vertical and radial drainage, U_v = average degree of consolidation for vertical drainage and U_r = degree of consolidation for radial drainage.

Recalling that

$$\bar{J}_v = m_v U_v \tag{45}$$

$$J_r = m_v U_r \tag{46}$$

$$\bar{J}_{vr} = m_v U_{vr} \tag{47}$$

and considering Eq. (44), the following expression may be written

$$\bar{J}_{vr} = \bar{J}_v + J_r - \frac{\bar{J}_v J_r}{m_v} \tag{48}$$

or, taking into account Eqs. (17) and (36), then

$$\bar{J}_{vr}(t) = m_v \left[1 - \sum_0^\infty \frac{2}{M^2} e^{-\left(M^2 + \frac{2c_r H^2}{c_v f(n) r_e^2} \right) \frac{c_v t}{H^2}} \right] \tag{49}$$

Making

$$\theta = \frac{2c_r H^2}{c_v f(n) r_e^2} \tag{50}$$

and comparing Eqs. (49) and (17), it is apparent that the solution for combined drainage may be easily obtained from substituting $M^2 T_v$ by $(M^2 + \theta) T_v$ in the expressions derived for vertical drainage. Therefore, the following equations apply to combined vertical and radial drainage.

For $t_{k-1} \le t \le t_k$ $(k \le n)$

$$\begin{aligned} \bar{\varepsilon}(T_v) = {} & m_v \sum_1^{k-1} \left\{ \Delta q_i \left[1 - \frac{2}{T_{v_i} - T_{v_{(i-1)}}} \sum_0^\infty \frac{1}{M^2(M^2+\theta)} \times \right.\right. \\ & \left.\left. \left(1 - e^{-(M^2+\theta)(T_{v_i} - T_{v_{(i-1)}})} \right) \times e^{-(M^2+\theta)(T_v - T_{v_i})} \right] \right\} + \\ & m_v \Delta q_k \left[\frac{T_v - T_{v_{(k-1)}}}{T_{v_k} - T_{v_{(k-1)}}} - \frac{2}{(T_{v_k} - T_{v_{(k-1)}})} \sum_0^\infty \frac{1}{M^2(M^2+\theta)} \times \right. \\ & \left. \left(1 - e^{-(M^2+\theta)(T_v - T_{v_{(k-1)}})} \right) \right] \end{aligned} \tag{51}$$

and

$$\begin{aligned} U_{vr}(T_v) = {} & \sum_1^{k-1} \left\{ \rho_i \left[1 - \frac{2}{T_{v_i} - T_{v_{(i-1)}}} \sum_0^\infty \frac{1}{M^2(M^2+\theta)} \times \right.\right. \\ & \left.\left. \left(1 - e^{-(M^2+\theta)(T_{v_i} - T_{v_{(i-1)}})} \right) \times e^{-(M^2+\theta)(T_v - T_{v_i})} \right] \right\} + \\ & \rho_k \left[\frac{T_v - T_{v_{(k-1)}}}{T_{v_k} - T_{v_{(k-1)}}} - \frac{2}{T_{v_k} - T_{v_{(k-1)}}} \sum_0^\infty \frac{1}{M^2(M^2+\theta)} \times \right. \\ & \left. \left(1 - e^{-(M^2+\theta)(T_v - T_{v_{(k-1)}})} \right) \right] \end{aligned} \tag{52}$$

where

$$\rho_i = \frac{\Delta q_i}{\sum_i^n \Delta q_i} \tag{25}$$

For $t > t_n$

$$\begin{aligned} \bar{\varepsilon}(T_v) = {} & m_v \sum_1^n \left\{ \Delta q_i \left[1 - \frac{2}{T_{v_i} - T_{v_{(i-1)}}} \sum_0^\infty \frac{1}{M^2(M^2+\theta)} \times \right.\right. \\ & \left.\left. \left(1 - e^{-(M^2+\theta)(T_{v_i} - T_{v_{(i-1)}})} \right) \times e^{-(M^2+\theta)(T_v - T_{v_i})} \right] \right\} \end{aligned} \tag{53}$$

and

$$\begin{aligned} U_{vr}(T_v) = {} & \sum_1^n \left\{ \rho_i \left[1 - \frac{2}{T_{v_i} - T_{v_{(i-1)}}} \sum_0^\infty \frac{1}{M^2(M^2+\theta)} \times \right.\right. \\ & \left.\left. \left(1 - e^{-(M^2+\theta)(T_{v_i} - T_{v_{(i-1)}})} \right) \times e^{-(M^2+\theta)(T_v - T_{v_i})} \right] \right\} \end{aligned} \tag{54}$$

Equations (23), (24), (28), (29), (40), (41), (42) and (43) are consistent with corresponding solutions presented in technical literature (Terzaghi and Frölich, 1936; Taylor, 1942; Schiffman, 1958; Olson, 1977; Kurma Rao and Vijaya Rama Raju, 1990; Da Mota, 1996; Lekha *et al.*, 1998), for one linear variable loading, since one makes $k = 1$ in Eqs. (23), (24), (40) and (41), vanishing the first term, and $n = 1$ in Eqs. (28) (29), (42) and (43).3.4.

A case solution

In order to present an application to the equations derived in this section, a consolidation analysis of a clay layer submitted to the loading history shown in Fig. 4 has been performed.

The loading features are

- 3 equal steps of loading with constant rate $\Delta q/\Delta t$
- 2 equal resting intervals of time between loadings with length $k\Delta t$

The analysis considers the three cases studied

- Only vertical drainage for $T_{v_1} = 0.1$; $k = 5$, 10 and 20

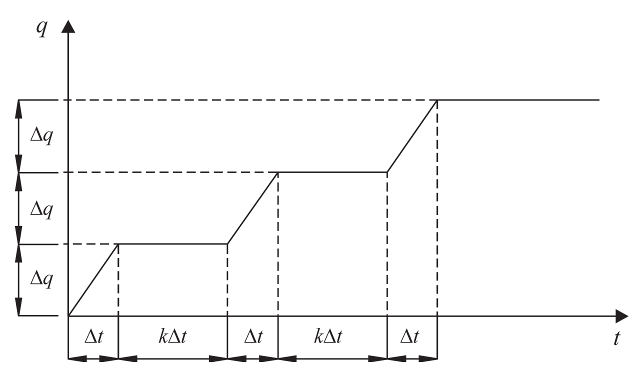

Figure 4 - Particular loading history considered in the application studied (linear variable three-step loading).

- Only radial drainage for $T_{r_1} = 0.1$; $k = 5$, 10 and 20; $n = 10$ and 30
- Combined vertical and radial drainage for $T_{v_1} = 0.1$; $k = 5$, 10 and 20; $\theta = 10$ and 100

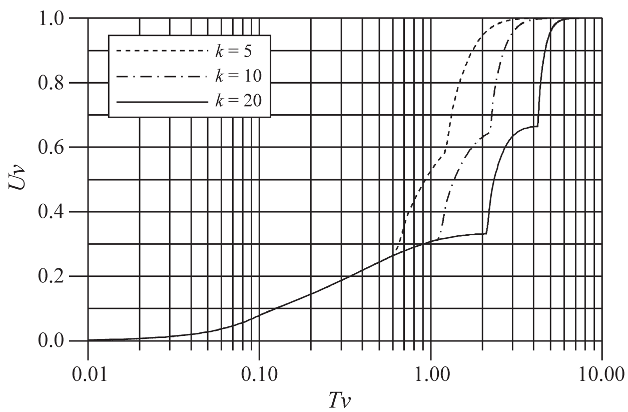

Figure 5 - Curves U_v x T_v for one-dimensional consolidation with vertical drainage under linear variable three-step loading and $T_{v_1} = 0.1$.

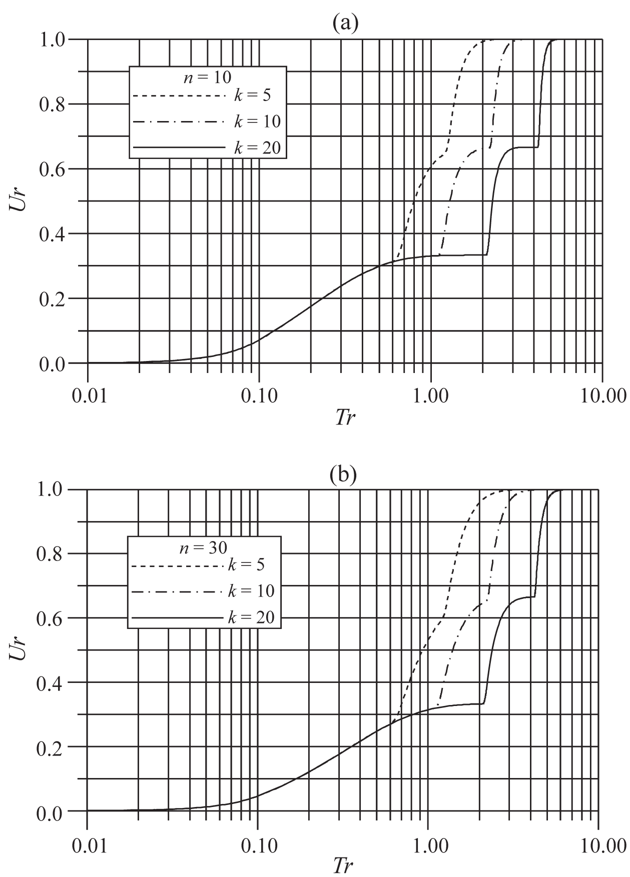

Figure 6 - Curves U_r x T_r for one-dimensional consolidation with radial drainage under linear variable three-step loading, $T_{r_1} = 0.1$, (a) $n = 10$ and (b) $n = 30$.

Figures 5 to Fig. 7 illustrate the results obtained. Consolidation for the second and third increments of load exhibits a steeper slope in the curves owing to the logarithmic scale. Values of U_r for $n = 10$ (Fig. 6a) increases faster than for $n = 30$ (Fig. 6b) because of the higher density of vertical drains in the former case. It can also be noticed that the progress of consolidation illustrated in Fig. 7(b) is faster than in Fig. 7(a). This behaviour can be explained by the magnitude of the parameter θ, representing the relative importance between vertical and radial drainage in combined conditions. Low values of θ indicate that vertical drainage plays a major role in consolidation. On the other hand, high values of θ mean that radial drainage is prevailing.

4. Second Application: Analysis of One-Dimensional Consolidation of Soft Soils under Embankment Loading With Partial Submersion of the Fill

When an embankment is constructed on the surface of a clay deposit with high water table level, its self-weight decreases with time owing to the partial submersion of the fill caused by the settlements. Therefore, consolidation analysis in this case must take into account the resulting variable de-

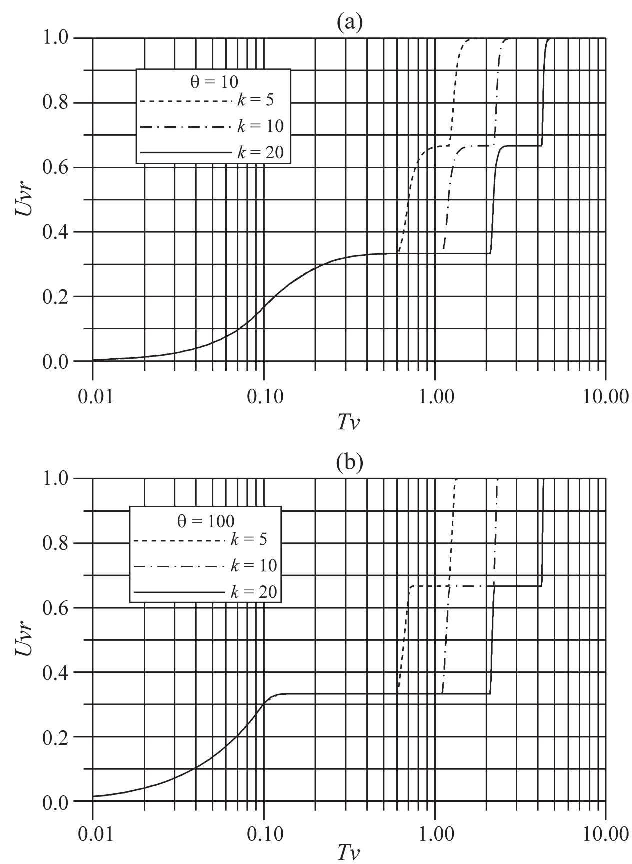

Figure 7 - Curves U_{vr} x T_v for one-dimensional consolidation with combined vertical and radial drainage under linear variable three-step loading, $T_{v_1} = 0.1$, (a) $\theta = 10$ and (b) $\theta = 100$.

creasing loading. Linear Viscoelasticity theory provides the necessary background for this analysis regarding the same basic assumptions considered in the previous application.

4.1. First case: vertical drainage

If a fill of height h and unit weight γ_t is placed onto a homogeneous deposit of soft clay of thickness H, the initial loading on the clay layer is (Fig. 8a)

$$q_0 = \gamma_t \times h \tag{55}$$

After an interval of time t, supposing for simplicity the water table located at the surface of the clay layer[2], the loading is (Fig. 8b).

$$q(t) = q_0 - \Delta\gamma H \bar{\varepsilon}(t) \tag{56}$$

where

$$\Delta\gamma = \gamma_t - \gamma_b, \tag{57}$$

γ_b = submerged (buoyant) unit weight of the fill.

Recalling that

$$\bar{\varepsilon}(t) = q(t)\bar{J}_v(0) - \int_0^t q(\tau)\frac{\partial \bar{J}_v(t-\tau)}{\partial\tau}d\tau \tag{19}$$

and substituting Eq. (56) into Eq. (19), yields

$$\begin{aligned}\bar{\varepsilon}(t) &= \left[q_0 - \Delta\gamma H\bar{\varepsilon}(t)\right]\bar{J}_v(0) - \\ &\int_0^t \left[q_0 H(\tau) - \Delta\gamma H\bar{\varepsilon}(\tau)\right]\frac{\partial \bar{J}_v(t-\tau)}{\partial\tau}d\tau\end{aligned} \tag{58}$$

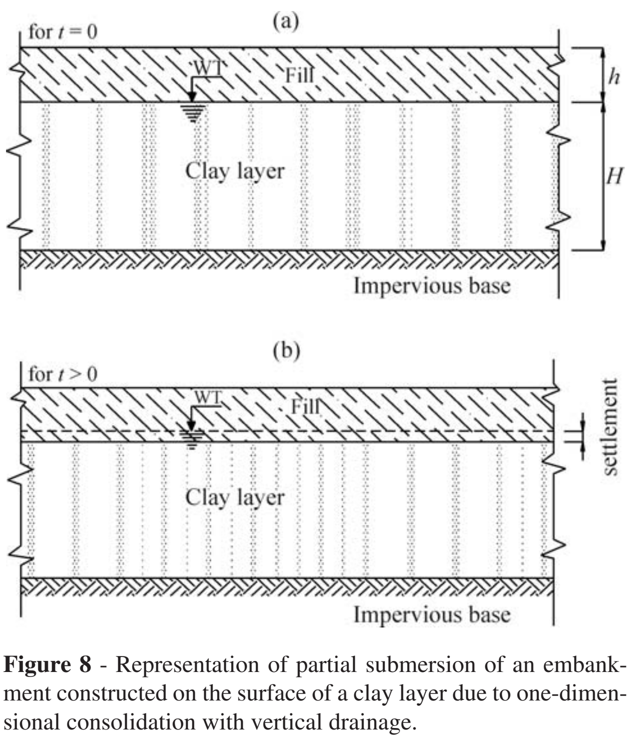

Figure 8 - Representation of partial submersion of an embankment constructed on the surface of a clay layer due to one-dimensional consolidation with vertical drainage.

An easier solution to Eq. (58) can be obtained if the approximate expressions for $U(T_v)$ are employed instead of the rigorous Terzaghi's solution, as follows

- early stages of consolidation $U_v = \left(\frac{4}{\pi}T_v\right)^{\frac{1}{2}}$ (59)

- late stages of consolidation $U_v = 1 - \frac{8}{\pi^2}e^{-\frac{\pi^2 T_v}{4}}$ (60)

4.1.1. Early stages of consolidation

From Eq. (59), it may be written

$$\bar{\varepsilon}(t) = m_v q_0 \left(\frac{4c_v t}{\pi H^2}\right)^{\frac{1}{2}} \tag{61}$$

and, therefore

$$\bar{J}_v(t) = m_v \left(\frac{4c_v t}{\pi H^2}\right)^{\frac{1}{2}} \tag{62}$$

Substituting Eq. (62) into Eq. (58), yields

$$\begin{aligned}\bar{\varepsilon}(t) = \frac{m_v}{H}\left(\frac{c_v}{\pi}\right)^{\frac{1}{2}}\Bigg[&q_0\int_0^t H(\tau)\times(t-\tau)^{-\frac{1}{2}}d\tau - \\ &\Delta\gamma H\int_0^t \bar{\varepsilon}(\tau)\times(t-\tau)^{-\frac{1}{2}}d\tau\Bigg]\end{aligned} \tag{63}$$

Applying Laplace transform to Eq. (63), resolving for $\hat{\bar{\varepsilon}}(s)$ and applying the inverse transform, results

$$\bar{\varepsilon}(T_v) = \frac{q_0 m_v}{\alpha}\left[1 - e^{\alpha^2 T_v}\times erfc\left(\alpha T_v^{\frac{1}{2}}\right)\right] \tag{64}$$

and

$$U_v(T_v) = \frac{1+\alpha}{\alpha}\left[1 - e^{\alpha^2 T_v}\times erfc\left(\alpha T_v^{\frac{1}{2}}\right)\right] \tag{65}$$

where

$$\alpha = m_v \Delta\gamma H \tag{66}$$

erfc ( ) = complementary error function

4.1.2. Late stages of consolidation

It may be inferred from Eq. (60) that

$$\bar{\varepsilon}(t) = m_v q_0 \left(1 - \frac{8}{\pi^2}e^{-\frac{\pi^2 c_v}{4H^2}t}\right) \tag{67}$$

and, therefore

2 Actually, the water table can be at any location. If, for instance, it is above the clay surface, q_0 should be conveniently calculated considering its initial partial submersion. On the other hand, if it is below the clay surface, $\Delta\gamma$ should take into account what soil is going to be submerged (clay only or clay and fill).

$$\bar{J}_v(t) = m_v\left(1-\frac{8}{\pi^2}e^{-\frac{\pi^2 c_v}{4H^2}t}\right) \quad (68)$$

Substituting Eq. (68) into Eq. (58), yields

$$\bar{\varepsilon}(t) = [q_0 - \Delta\gamma H\bar{\varepsilon}(t)]\times\left[m_v\left(1-\frac{8}{\pi^2}\right)\right] + \frac{2m_v c_v}{H^2}\times \left[q_0\int_0^t H(\tau)e^{-\frac{\pi^2 c_v}{4H^2}(t-\tau)}d\tau - \int_0^t \bar{\varepsilon}(\tau)e^{-\frac{\pi^2 c_v}{4H^2}(t-\tau)}d\tau\right] \quad (69)$$

Applying Laplace transform to Eq. (69), resolving for $\hat{\bar{\varepsilon}}(s)$ and finding the inverse transform, yields

$$\bar{\varepsilon}(T_v) = \frac{q_0 m_v}{(1+\alpha)}\left[1-\frac{8}{\pi^2}\frac{1}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}e^{-\frac{1+\alpha}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}\frac{\pi^2}{4}T_v}\right] \quad (70)$$

and the average degree of consolidation

$$U_v(T_v) = 1-\frac{8}{\pi^2}\frac{1}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}e^{-\frac{1+\alpha}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}\frac{\pi^2}{4}T_v} \quad (71)$$

For $T_v \to \infty$, Eq. (70) becomes

$$\bar{\varepsilon}(\infty) = \frac{q_0 m_v}{(1+\alpha)} = \frac{\bar{\varepsilon}(T_v \to \infty,\ \text{without submersion})}{1+\alpha} \quad (72)$$

Considering the logarithmic relationship between void ratio and vertical effective stress,

$$m_v = \frac{C_c}{q_0(1+e_0)}\log\left(\frac{p'_0+q_0}{p'_0}\right) \quad (73)$$

where e_0 = initial void ratio, p'_0 = initial effective stress and C_c = compression index, the parameter α may be evaluated from

$$\alpha = \frac{\Delta\gamma H}{q_0}\times\frac{C_c}{1+e_0}\times\log\left(\frac{p'_0+q_0}{p'_0}\right) \quad (74)$$

It can be observed that Eqs. (65) and (71), for situations (4.1.1) and (4.1.2) respectively, assume the same value for T_v approximately equal to $\frac{0.213}{1+3\alpha/2}$. Therefore, the following may be stated:

For $T_v \le \frac{0.213}{1+3\alpha/2}$, use equations

$$\bar{\varepsilon}(T_v) = \frac{q_0 m_v}{\alpha}\left[1-e^{\alpha^2 T_v}\times erfc\left(\alpha T_v^{\frac{1}{2}}\right)\right] \quad (64)$$

$$U_v T_v = \frac{1+\alpha}{\alpha}\left[1-e^{\alpha^2 T_v}\times erfc\left(\alpha T_v^{\frac{1}{2}}\right)\right] \quad (65)$$

For $T_v \ge \frac{0.213}{1+3\alpha/2}$ use equations

$$\bar{\varepsilon}(T_v) = \frac{q_0 m_v}{(1+\alpha)}\left[1-\frac{8}{\pi^2}\frac{1}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}e^{-\frac{1+\alpha}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}\frac{\pi^2}{4}T_v}\right] \quad (70)$$

$$U_v(T_v) = 1-\frac{8}{\pi^2}\frac{1}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}e^{-\frac{1+\alpha}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}\frac{\pi^2}{4}T_v} \quad (71)$$

Curves U_v x T_v for α = 0.20, 0.50 and 0.80 and T_v x α for U_v = 50%, 70% and 90% are shown in Figs. 9 and 10.

4.2. Second case: radial drainage

Now suppose a fill of height h and unit weight γ_t placed onto the surface of a homogeneous deposit of soft

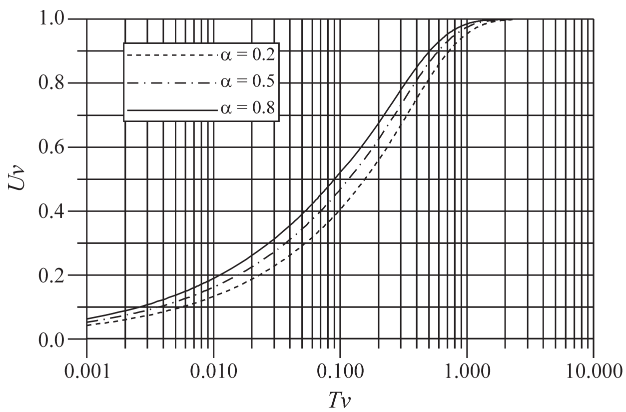

Figure 9 - Curves U_v x T_v for vertical drainage with partial submersion of the fill.

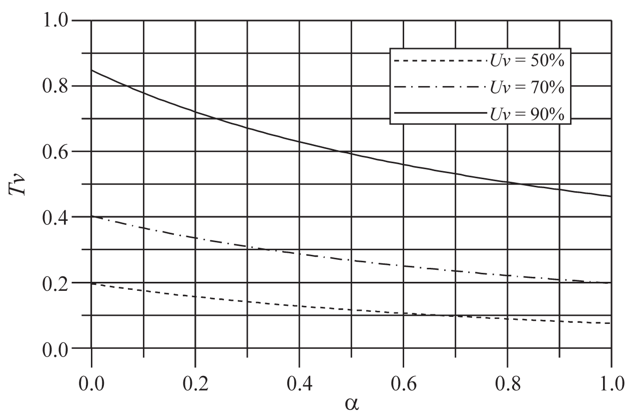

Figure 10 - Curves T_v x α for vertical drainage and partial submersion of the fill.

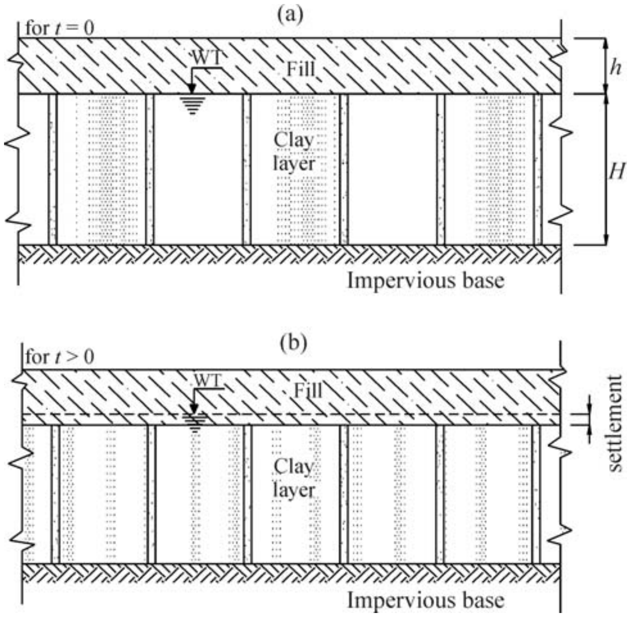

Figure 11 - Representation of partial submersion of an embankment constructed on the surface of a clay layer due to one-dimensional consolidation with radial drainage.

clay of thickness H, where vertical drains have been installed, assuming no vertical drainage occurring (Fig. 11). As in the first case (vertical drainage), the initial loading is

$$q_0 = \gamma_t \times h \tag{55}$$

and the loading after an interval of time t is

$$q(t) = q_0 - \Delta\gamma H \varepsilon(t) \tag{56a}$$

where $\Delta\gamma$ has the same meaning as in the first case.

Recalling that

$$\varepsilon(t) = q(t) J_r(0) - \int_0^t q(\tau) \frac{\partial J_r(t-\tau)}{\partial \tau} d\tau \tag{37}$$

and inserting Eq. (56a) into Eq. (37), results

$$\begin{aligned} \varepsilon(t) = {} & \left[q_0 - \Delta\gamma H \varepsilon(t)\right] J_r(0) - \\ & \int_0^t \left[q_0 H(\tau) - \Delta\gamma H \varepsilon(\tau)\right] \frac{\partial J_r(t-\tau)}{\partial \tau} d\tau \end{aligned} \tag{75}$$

Substituting Eq. (36) (creep function for radial drainage) into Eq. (75), applying Laplace transform, resolving for $\hat{\varepsilon}(s)$ and finding the inverse transform, yields

$$\varepsilon(T_r) = \frac{q_0 m_v}{(1+\alpha)} \left[1 - e^{-\frac{8T_r(1+\alpha)}{f(n)}}\right] \tag{76}$$

and the degree of consolidation is

$$U_r(T_r) = 1 - e^{-\frac{8T_r(1+\alpha)}{f(n)}} \tag{77}$$

Similarly to the vertical drainage case, the final strain is

$$\varepsilon(\infty) = \frac{q_0 m_v}{(1+\alpha)} = \frac{\varepsilon(T_r \to \infty, \text{without submersion})}{1+\alpha} \tag{78}$$

Figures 12 and 13 present respectively the curves U_r x T_r for $n = 10$ and 30 and $\alpha = 0.20$, 0.50 and 0.80, and T_r x α for $n = 10$ and 30 and $U_r = 50\%$, 70% and 90%.

4.3. Third case: combined vertical and radial drainage

Solution for combined vertical and radial drainage regarding the early stages of consolidation cannot be easily derived. However, a straightforward approximate solution can be obtained for the late stages, as follows.

Using Carrillo's expression (44), it has been shown that

$$\bar{J}_{vr} = \bar{J}_v + J_r - \frac{\bar{J}_v J_r}{m_v} \tag{48}$$

Substituting Eqs. (68) and (36) into Eq. (48), results

$$\bar{J}_{vr}(t) = m_v \left(1 - \frac{8}{\pi^2} e^{-\left(\frac{\pi^2}{4}+\theta\right)\frac{c_v t}{H^2}}\right) \tag{79}$$

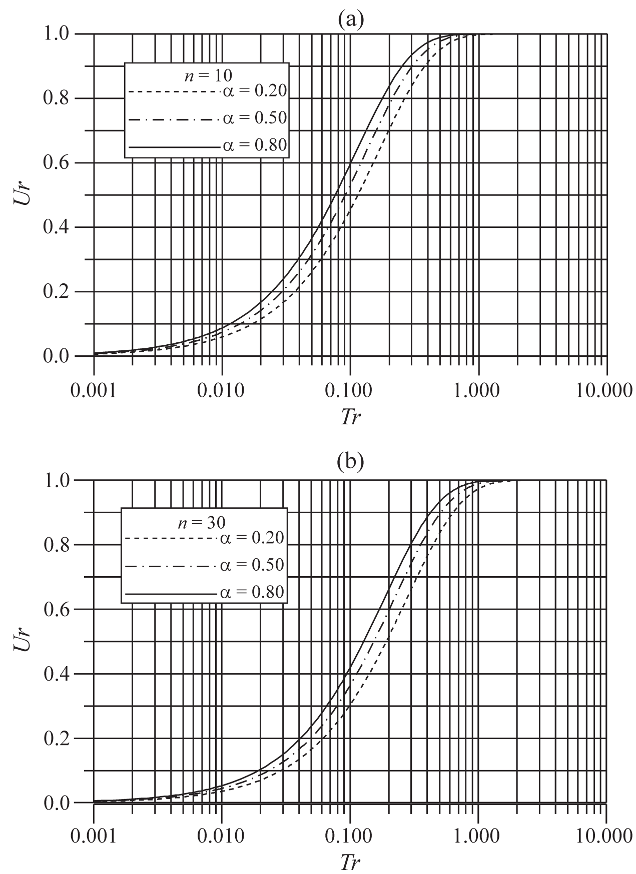

Figure 12 - Curves U_r x T_r for radial drainage with partial submersion of the fill, (a) $n = 10$ and (b) $n = 30$.

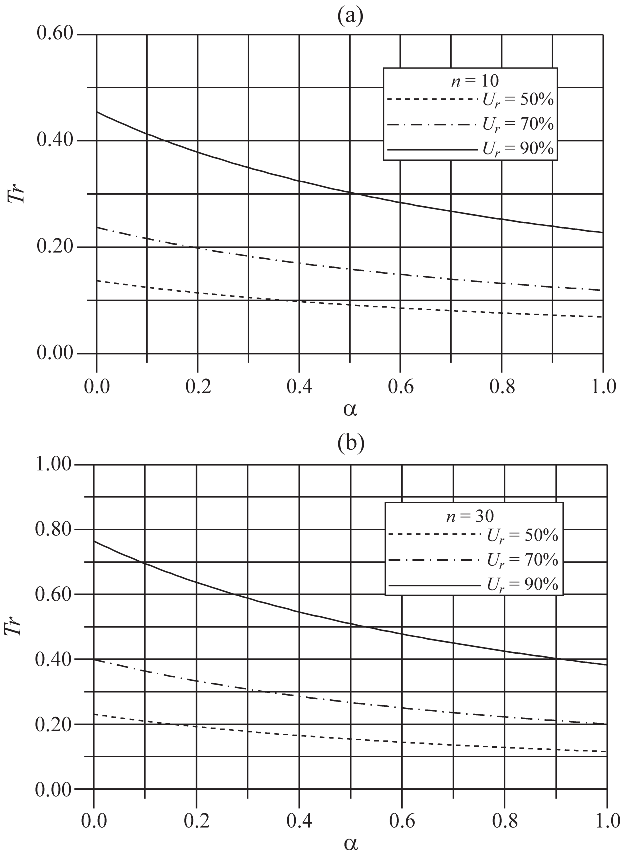

Figure 13 - Curves T_r x α for radial drainage and partial submersion of the fill, (a) n = 10 and (b) n = 30.

It can be written

$$\bar{\varepsilon}(t) = \left[q_0 - \Delta\gamma H\bar{\varepsilon}(t)\right]\bar{J}_{vr}(0) - \int_0^t \left[q_0 H(\tau) - \Delta\gamma H\bar{\varepsilon}(\tau)\right]\frac{\partial \bar{J}_{vr}(t-\tau)}{\partial\tau}\,d\tau \tag{80}$$

Substituting Eq. (79) into Eq. (80), applying Laplace transformation, resolving for $\hat{\bar{\varepsilon}}(s)$ and finding the inverse transform, yields

$$\bar{\varepsilon}(T_v) = \frac{q_0 m_v}{(1+\alpha)}\left[1 - \frac{8}{\pi^2}\frac{1}{1+\alpha\left(1-\frac{8}{\pi^2}\right)} \times e^{-\frac{1+\alpha}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}\left(\frac{\pi^2}{4}+\theta\right)T_v}\right] \tag{81}$$

and

$$U_{vr}(T_v) = 1 - \frac{8}{\pi^2}\frac{1}{1+\alpha\left(1-\frac{8}{\pi^2}\right)} \times e^{-\frac{1+\alpha}{1+\alpha\left(1-\frac{8}{\pi^2}\right)}\left(\frac{\pi^2}{4}+\theta\right)T_v} \tag{82}$$

which hold for

$$T_v \geq \frac{0.213}{1+\frac{3\alpha}{2}} \tag{83}$$

Naturally, for $T_v \to \infty$, Eq. (81) gives

$$\bar{\varepsilon}(\infty) = \frac{q_0 m_v}{(1+\alpha)} = \frac{\bar{\varepsilon}(T_v \to \infty, \text{without submersion})}{1+\alpha} \tag{72}$$

Figure 14 illustrates the results of combined drainage through the curves U_{vr} x T_v for α = 0.20, 0.50 and 0.80 and θ = 10.

4.4. Final remark on the second application

The non-dimensional parameter α varies mainly between 0 and 1 and affects both the amount of settlement and the consolidation rate, as can be seen from Eqs. (64), (70), (76) and (81). Diagrams of Figs. 9-10 and Figs. 12-14 show how α affects the progress of consolidation. However, although larger α implies faster consolidation in terms of time factor, when vertical drainage takes place this conclusion may not hold for real time if one compares deposits of different thicknesses.

5. Third application: Analysis of One-Dimensional Consolidation of Soft Soils by Drain Columns of Finite Stiffness

Installation of drain wells in soft clay deposits aiming the acceleration of the consolidation process is a technique ordinarily employed in design of embankments on soft soils. Although the use of flexible pre-fabricated drain poses generally economic advantages, there are particular situations where the availability of low cost sand or stone nearby the work site allows the utilization of these materials as drain columns. Besides, drain columns of finite vertical stiffness also reduce the final settlement, becoming thus an economically attractive solution in some cases.

The purpose of this application is to solve the problem of the one-dimensional consolidation of a cell comprised by a cylinder of soil having a diameter d_e surround-

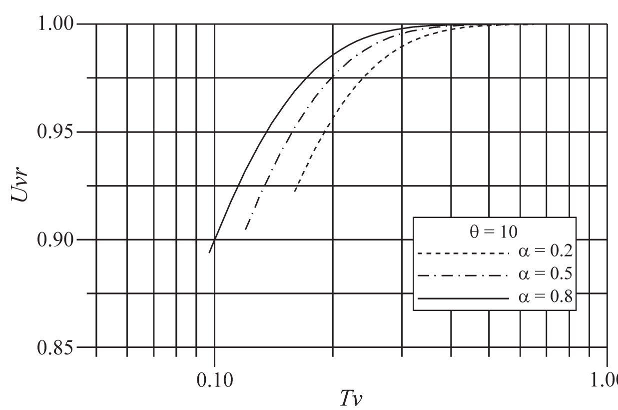

Figure 14 - Curves U_{vr} x T_v for combined vertical and radial drainage with partial submersion of the fill and θ = 10.

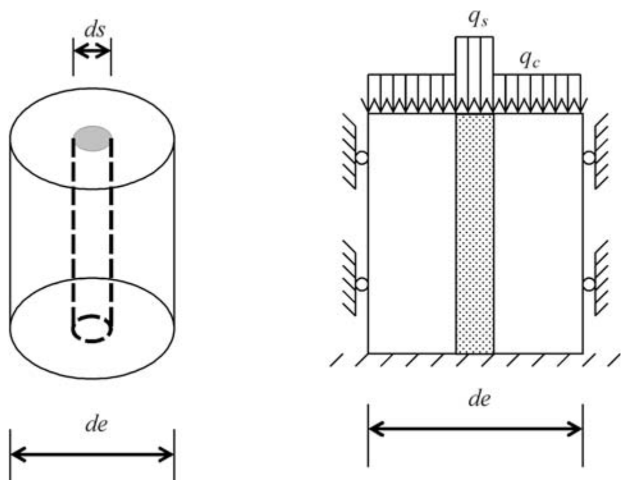

Figure 15 - Representation of a cylinder of soil surrounding a stiff drain column (after Barksdale and Bachus, 1983, modified).

ing a stiff drain column having a diameter d_s (Fig. 15). In most of the practical problems the influence of the vertical drainage may be neglected when compared to the radial drainage and, therefore, only this latter condition is considered herein.

Besides the assumptions regarding the behaviour of the clay considered in the first application presented, the following assumptions are also assumed:

(6) the drain column material has finite stiffness with linear stress-strain relationship;

(7) the lateral displacements of the drain column are very small, not affecting therefore the consolidation of the clay;

(8) the clay layer and the drain column have the same strain at any time after loading (equal strain).

Taking into account the high values of vertical strains normally associated with problems of embankments on soft soils, one can assume that the drain column is under failure condition in most of its length. However, as the radial compressive stresses acting on the column also increase with the strain, owing to the increase in the effective stresses in the clay layer, it is reasonable to admit, as an approximation, the linear stress-strain behaviour stated in the assumption (6).

According to Barksdale and Bachus (1983), area replacement ratio, a_s, is defined as

$$a_s = \frac{A_s}{A} \tag{84}$$

where A_s is the area of the drain column and A is the total area within the cell. The ratio of the area of the soil remaining, A_c, to the total area A is then

$$a_c = 1 - a_s \tag{85}$$

The area replacement ratio, a_s, may also be expressed as a function of the diameter and spacing of the drain columns by the following equation

$$a_s = C_1 \left(\frac{d_s}{s} \right)^2 \tag{86}$$

where d_s = diameter of the drain column, s = centre-to-centre spacing of the drain columns and C_1 = constant dependent upon the pattern of drain columns used; for a square pattern $C_1 = \pi/4$; for equilateral triangular pattern $C_1 = \pi / (2\sqrt{3})$, or

$$a_s = \frac{1}{n^2} \tag{87}$$

Stress concentration factor, ν, is defined as

$$\nu = \frac{q_s}{q_c} \tag{88}$$

where q_s = vertical loading stress acting on the top of the drain column and q_c = vertical loading stress acting on the surface of the clay layer.

The mean vertical loading stress q on the top of the cell can be obtained by equilibrium condition, as follows

$$q = q_s \times a_s + q_c (1 - a_s) \tag{89}$$

Denoting K_s the modulus of deformation of the drain column, one may write for any time t

$$\varepsilon(t) = \frac{q_s(t)}{K_s} \tag{90}$$

Equating the strain in the clay layer (Eq. (37)) to the strain in the column (Eq. (90)) and applying Laplace transform to the resulting expression and also to Eq. (89), yields

$$\begin{cases} \dfrac{\hat{q}_s(s)}{K_s} = \dfrac{2c_r}{r_e^2 f(n)} \times \dfrac{1}{s + \dfrac{2c_r}{r_e^2 f(n)}} \times \hat{q}_c(s) \\ \dfrac{q}{s} = \hat{q}_s(s) \times a_s + \hat{q}_c(s) \times (1 - a_s) \end{cases} \tag{91}$$

Resolving the system of Eqs. (91) and finding the inverse transform, yields

$$\frac{q_c(T_r)}{q} = \frac{1}{1 - a_s} \left[\frac{1 - a_s + \beta \times e^{-8 \frac{(1 - a_s + \beta)}{(1 - a_s) f(n)} T_r}}{1 - a_s + \beta} \right] \tag{92}$$

$$\frac{q_s(T_r)}{q} = \frac{1}{a_s} \left[1 - \frac{1 - a_s + \beta \times e^{-8 \frac{(1 - a_s + \beta)}{(1 - a_s) f(n)} T_r}}{1 - a_s + \beta} \right] \tag{93}$$

where

$$\beta = a_s K_s m_v \tag{94}$$

The stress concentration factor, ν, can be obtained from Eqs. (88), (92) and (93).

The vertical strain in the soil layer can be obtained substituting Eqs. (36) and (92) into Eq. (37), which may be solved using Laplace transforms, giving

$$\varepsilon(T_r) = q \frac{m_v}{1 - a_s + \beta}\left(1 - e^{-8\frac{(1-a_s+\beta)}{(1-a_s)f(n)}T_r}\right) \tag{95}$$

The average degree of consolidation is, then

$$U(T_r) = 1 - e^{-8\frac{(1-a_s+\beta)}{(1-a_s)f(n)}T_r} \tag{96}$$

The parameter β represents the relative stiffness between the drain column and a fictitious column of soil having the diameter d_e, previously defined. The progress of the concentration factor with time can be seen in Fig. 16 for a particular case of $a_s = 0.02$ and $\beta = 0.2$, evidencing the gradual and partial unloading of the soil, shifting the embankment weight from the clay layer to the drain columns.

The parameter β varies in the range 0-∞. It is worth discussing the above equations for two particular values of β, corresponding to $K_s = 0$ and $K_s = 1/m_v$.

(1) $\beta = 0$ (for $K_s = 0$)

In this case, $\frac{q_c}{q} = \frac{1}{1-a_s}$ and $\frac{q_s}{q} = 0$, representing the consolidation of a clay layer under a constant loading $\frac{q}{1-a_s}$.

(2) $\beta = a_s$ (for $K_s = 1/m_v$)

In this case, clay and drain have the same compressibility. Since the drain behaves as an elastic-instantaneous material (*i.e.*, it has no time-dependent constitutive equations), at the initial time ($t = 0$) there is no vertical strain in the clay and therefore all the loading is borne by the clay. As consolidation progresses, part of the loading bore by the clay is shifted to the drain column and eventually, at the end of consolidation ($t \to \infty$), the loading stress on the clay is the same as on the drain.

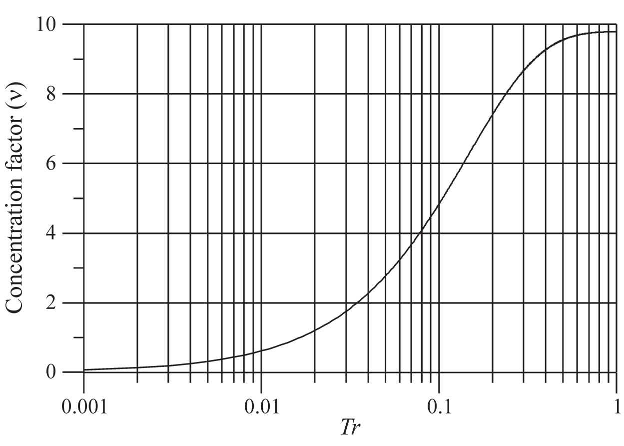

Figure 16 - Progress of the concentration factor (ν) with T_r for $a_s = 0.02$ and $\beta = 0.2$.

6. Conclusions

The Linear Viscoelasticity theory is a powerful tool to solve one-dimensional consolidation problems under variable loading. Although primary consolidation is strictly a hydrodynamic phenomenon, it may be successfully treated as a viscoelastic problem in terms of total stresses. This approach is similar to considering saturated clay soil exhibiting Tresca yield envelope when submitted to undrained loading, in terms of total stresses, when its plastic behaviour is analysed. This first and certainly the main conclusion of this work resulted from the straightforward way that three study cases involving consolidation of soft soils under variable loading were solved. The closed form solutions obtained are also relevant for both design practice and validation of numerical models.

The expressions obtained for one-dimensional consolidation under a number of linear variable loads are applicable to any loading history prescribed as long as it can be subdivided into several increments of load. It is worth mentioning that it may be very useful in embankment design to plan the most suitable loading history to achieve pre-set degrees of consolidation at particular times.

The diagrams *U* x *T* produced from the solutions allow a general overview of the progress of consolidation for a loading programme of three ramp loadings with two resting intervals. They also evidence how parameters *n* (radial drainage) and θ (combined drainage) affects the consolidation, although the logarithmic scale attenuates the differences.

The Viscoelasticity theory provides closed form solutions for the problem of one-dimensional consolidation of a deposit of clay under embankment loading when partial submersion of the fill occurs. As far as vertical drainage is concerned, classical approximate solutions were employed to obtain the corresponding creep function. Thus, in this case, two different expressions were derived for early and late stages of consolidation. For radial drainage, however, an exact solution was obtained from Barron's equation. For combined vertical and radial drainage only one expression was derived, regarding late stages of consolidation.

The non-dimensional parameter α is quite important in the analysis of submersion since it affects not only the amount of final settlement, but also the consolidation rate. In general, larger α implies faster consolidation, although this may not be true when vertical consolidation takes place.

The consolidation of a clay layer with drain columns of finite stiffness is also a variable loading problem easily tackled by Linear Viscoelasticity. The solution provides equations to determine stresses and strain on the soil and columns and the average degree of consolidation at any time taking into account the area replacement ratio and the modulus of deformation of the drain column.

The parameter β represents the relative stiffness between the drain column and a fictitious column of soil, influencing both the amount of settlement and the consolidation rate.

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List of Symbols

a_s = area replacement ratio
c_r = coefficient of consolidation for radial flow
c_v = coefficient of consolidation for vertical flow
C_1 = constant dependent upon the pattern of drain columns used
C_c = compression index
d_e = diameter of zone of influence of the soil mass
d_s = diameter of the drain column
e_0 = initial void ratio
erfc ( ) = complementary error function
h = height of an embankment constructed on the surface of a clay deposit
H = thickness of a clay layer
$H(t)$ = Heaviside function
$J(t)$ = creep function
$J_r(t)$ = creep function for one-dimensional consolidation with radial drainage
$\bar{J}_v(t)$ = average creep function for one-dimensional consolidation with vertical drainage
$\bar{J}_{vr}(t)$ = average creep function for one-dimensional consolidation with combined vertical and radial drainage
K_s = modulus of deformation of the drain column
m_v = coefficient of volume compressibility
n = ratio between radius of the zone of influence of a vertical drain and its radius
p'_0 = initial effective stress
q_0 = inicial surcharge on the surface of a clay layer
q_c = vertical loading stress acting on the surface of the clay layer
q_s = vertical loading stress acting on the top of the drain column
r_d = radius of a vertical drain
r_e = radius of the zone of influence of a vertical drain
$R(t)$ = relaxation function
s = centre-to-centre spacing of the drain columns
s = auxiliar variable in the Laplace transformation
$s(t)$ = settlement of the top of a clay layer at time t
$s(\infty)$ = final settlement of the top of a clay layer, at infinite time
t = time
T_r = time factor for one-dimensional consolidation with radial drainage
T_v = time factor for one-dimensional consolidation with vertical drainage
U_r = degree of consolidation for one-dimensional consolidation with radial drainage
U_v = average degree of consolidation for one-dimensional consolidation with vertical drainage
U_{vr} = average degree of consolidation for one-dimensional consolidation with combined vertical and radial drainage
$\alpha = m_v \Delta\gamma H$ (non-dimensional parameter related to consolidation with partial submersion of the fill)
$\beta = a_s k_s m_v$ (non-dimensional parameter related to consolidation with drain columns)
$\hat{\bar{\varepsilon}}(s)$ = Laplace transform of $\bar{\varepsilon}(t)$
$\bar{\varepsilon}(t)$ = average vertical strain at time t
$\bar{\varepsilon}(\infty)$ = final average vertical strain, at infinite time
γ_b = submerged unit weight of a fill
γ_t = unit weight of a fill
ν = stress concentration factor
$\theta = \dfrac{2c_r H^2}{c_v f(n) r_e^2}$ (non-dimensional parameter related to consolidation with combined radial and vertical drainage)
ρ = ratio between increment of load and total load applied
σ_0 = total stress applied, constant with time