Explicit Numerial Iterative Methods Applied to the Three-Parameter Infiltration Equation

A.L.B. Cavalcante, L.C.S.M. Ozelim, P.K. Swamee, P.N. Rathie

Abstract. Finding explicit solutions to the partial differential equations governing the infiltration phenomenon tend to be a challenging issue. On account of such complexity, researchers have proposed algebraic equations to model this phenomenon. This paper deals with the solution of the general three-parameter infiltration equation, which is an implicit equation corresponding to the interpolation between the Green-Ampt and Talsma-Parlange infiltration models. By means of the Householders method, the cumulative infiltration is obtained by a rapidly converging iterative procedure. The results are further manipulated in order to obtain the infiltration rate explicitly. The effect of variables involved on the solution is also discussed. A case study is included to demonstrate the use of the proposed algorithm. The formulas proposed provide easy-to-use equations which enable calculations on the fly while considering practical applications.

Keywords: three-parameter infiltration formula, Green-Ampt, Householder's Methods, Talsma-Parlange.

1. Introduction

Originally proposed by Parlange *et al.* (1982), the three parameter infiltration model unifies two other well-known models, introduced by Green-Ampt (1911) and Talsma & Parlange (1972). Green-Ampt's proposition (Green & Ampt, 1911) figures as one of the most explored infiltration models. Green-Ampt's model is based on Darcy's equation and a few other assumptions, namely: the existence of a constant hydraulic pressure head at soil's surface; the hydraulic conductivity in such zone K_t is equivalent to the saturated one K_s; and the formation of a precisely defined wetting front leading to the so-called piston movement (Zonta *et al.* 2010).

The integrated form of Green-Ampt infiltration rate equation is given by Mein & Farrel (1974):

$$t_* = I_* - \ln(1 + I_*) \tag{1}$$

where t_* = nondimensional time; and I_* = nondimensional cumulative infiltration, given by (Mein & Farrel, 1974):

$$t_* = \frac{K_s(t - t_p + t_s)}{(\psi_w - p_s)(\theta_s - \theta_o)} \tag{2}$$

$$I_* = \frac{I}{(\psi_w - p_s)(\theta_s - \theta_o)} \tag{3}$$

wherein I is the cumulative infiltration; t_p = ponding time; t_s = theoretical time necessary to infiltrate the same volume infiltrated until the ponding time, under saturation of the superficial layer of the soil; ψ_w = average wetting front matric potential; p_s = average surface pressure head and θ_s = saturated water content; θ_o = initial water content.

On the other hand Talsma & Parlange (1972) equation for the nondimensional cumulative infiltration and nondimensional time is:

$$(I_* - t_* - 1)\exp(I_* - t_* - 1) + \exp(-t_* - 1) = 0 \tag{4}$$

Equations 1 and 4 were unified by Parlange *et al.* (1982) as

$$t_* = I_* + (1-\alpha)^{-1} \ln\left[\frac{\alpha}{1 - (1-\alpha)\exp(-\alpha I_*)}\right] \tag{5}$$

where α = a transition parameter lying in the range $0 \le \alpha \le 1$. When $\alpha = 0$, the Green-Ampt equation is recovered from Eq. 5. On the other hand, when $\alpha = 1$, Talsma-Parlange's model follows from Eq. 5.

It has been shown by Parlange *et al.* (2002) that both the Green-Ampt and Talsma-Parlange equations can be explicitly solved in terms of Lambert W-function. On the other hand, Rathie *et al.* (2012) proposed the explicit solution of Eq. 5 in terms of the H-function. When implementations of such special functions are not available, standard trial and error procedures are used to solve Eq. 5.

Practical situations require neither complex nor precisely defined exact solutions. They do demand accurate solutions. This paper presents approximate solutions to the tree-parameter infiltration equation, by means of an iterative algorithm solution whose single iteration order of convergence can be chosen as high as necessary.

André L. Brasil Cavalcante, PhD, Associate Professor, Departamento de Engenharia Civil e Ambiental, Universidade de Brasília, 70910-900 Brasília, DF, Brazil. e-mail: abrasil@unb.br.
Luan Carlos de S.M. Ozelim, PhD Student, Departamento de Engenharia Civil e Ambiental, Universidade de Brasília, 70910-900 Brasília, DF, Brazil. e-mail: luanoz@gmail.com.
Prabhata K. Swamee, PhD, Full Professor, Department of Civil Engineering, National Institute of Technology, 144 011 Jalandhar, India. e-mail: swameefce@yahoo.com.
Pushpa N. Rathie, PhD, Senior Researcher, Departamento de Estatística, Universidade de Brasília, 70910-900 Brasília, DF, Brazil. e-mail: pushpanrathie@yahoo.com.
Submitted on August 12, 2011; Final Acceptance on April 19, 2013; Discussion open until August 30, 2013.

2. Solution of the Three-Parameter Infiltration Equation Using Householder's Method

Consider a function $f(x)$ whose root r needs to be determined. By choosing an initial point, x_0, Householder (1970) proposed the general iteration root-finding recurrence formula:

$$x_{n+1} = x_n + (a+1)\frac{[1/f(x_n)]^{(a)}}{[1/f(x_n)]^{(a+1)}} \tag{6}$$

in which a is a real number which governs the rate at which the iterative scheme tends to the root of the nonlinear equation $f(x) = 0$ and $(1/f)^{(a)}$ represents the a-th derivative of $1/f(x)$.

In fact, it can be shown that $|x_{n+1} - r| \le |K\, x_n - r|^{a+2}$ for some positive real number K, thus if x_0 is sufficiently close to r, the iteration scheme converges to the value of r with rate $a + 2$. Note that this method is a generalization of the Newton-Raphson method, which can be easily obtained by taking $a = 0$ in Eq. 6.

Consider that Eq. 3 can be rewritten as:

$$f(I_*) = t_* - I_* + (1-\alpha)^{-1}\ln\left[\frac{1-(1-\alpha)\exp(-\alpha I_*)}{\alpha}\right] \tag{7}$$

The first and second derivatives of $f(I_*)$ with respect to I_* are:

$$\frac{\partial}{\partial I_*} f(I_*) = -1 + \left[\frac{\alpha}{\exp(\alpha I_*)-(1-\alpha)}\right] \tag{8}$$

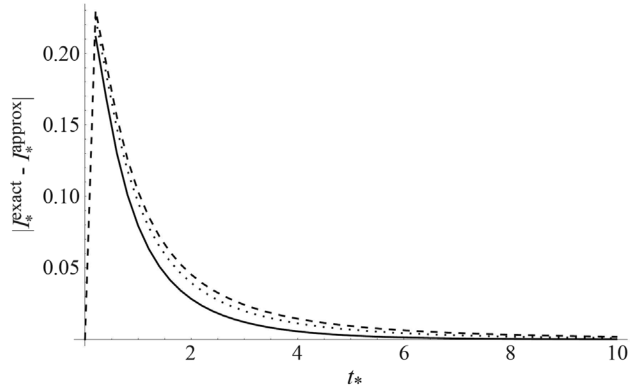

Figure 1 - Absolute Departure of Eq. 11 from exact solution of Eq. 5.

$$\frac{\partial^2}{\partial I_*^2} f(I_*) = \frac{-\alpha^2 \exp(\alpha I_*)}{[\exp(\alpha I_*)-(1-\alpha)]^2} \tag{9}$$

In order to use Houlseholder's iteration formula, let one establish as first guess for I_* the value of t_*. This comes from the fact that as I_* grows, Eq. 7 tends to $f(I_*) \approx t_* - I_*$. Using Eq. 6, I_* is found to be:

(a) For quadratic convergence of the first iteration ($a = 0$)

$$I_* = t_* - \frac{[\exp(\alpha t_*)-(1-\alpha)]\ln\left[\frac{1-(1-\alpha)\exp(-\alpha t_*)}{\alpha}\right]}{(1-\alpha)[1-\exp(\alpha t_*)]} \tag{10}$$

(b) For cubic convergence of the first iteration ($a = 1$)

$$I_* = t_* - \frac{2[1-\exp(\alpha t_*)][\exp(\alpha t_*)-(1-\alpha)]}{\alpha^2\exp(\alpha t_*) + 2[1-\exp(\alpha t_*)]^2\left\{(1-\alpha)^{-1}\ln\left[\frac{1-(1-\alpha)\exp(-\alpha t_*)}{\alpha}\right]\right\}} \tag{11}$$

It can be noticed that the solution is an iterative algorithm.

In order to compare Eq. 11 to the exact solution of Eq. 5, consider the absolute difference between both solutions. Such departure is shown in Fig. 1. The exact solution has been obtained by trial and error procedures.

Figure 1 reveals that Eq. 11 behaves similarly for different values of the interpolation value α. The latter figure also shows that the maximum departure happens at low values of the nondimensional time, which is fully justified by the "long term" initial guess for nondimensional cumulative infiltration.

It is worth noticing that one may find an optimal initial guess in order to diminish the absolute departure over the whole domain. Also, the results in Fig. 1 correspond to a single iteration of the algorithm. Thus, if more iterations where taken into account, the error would be mitigated. The methodology is, this way, verified to produce good estimations of the solutions.

3. Determination of Some of the Equation Parameters

In order to obtain the values of t_p and t_s, one shall consider the following equation, obtained by means of implicit differentiation of Eq. 5 with respect to t_*:

$$I = \frac{(\psi_w - p_s)(\theta_s - \theta_o)}{\alpha}\ln\left(1 + \frac{\alpha K_s}{r - K_s}\right) \tag{12}$$

in which r is the infiltration rate. If one wants to find the total infiltration until ponding time, I_p, one shall make the substitution $r = i_p$ in Eq. 12, thus

$$I_p = \frac{(\psi_w - p_s)(\theta_s - \theta_o)}{\alpha}\ln\left(1 + \frac{\alpha K_s}{i_p - K_s}\right) \tag{13}$$

where i_p is the rainfall intensity. It's simple to get t_p from Eq. 13 since $t_p = I_p/i_p$, which provides:

$$t_p = \frac{(\psi_w - p_s)(\theta_s - \theta_o)}{\alpha i_p} \ln\left(1 + \frac{\alpha K_s}{i_p - K_s}\right) \quad (14)$$

One shall note that when α tends to 0, the same relation obtained by Mein & Farrel (1974) is retrieved for a Green-Ampt case, which is:

$$t_p = \frac{(\psi_w - p_s)(\theta_s - \theta_o) K_s}{i_p (i_p - K_s)} \quad (15)$$

The value of t_s can be obtained as follows:

$$t_s = \frac{I_p}{K_s} + \frac{(1-\alpha)^{-1} (\psi_w - p_s)(\theta_s - \theta_o)}{K_s} \ln\left[\frac{\alpha}{1 - (1-\alpha)\exp(\alpha I_p)}\right] \quad (16)$$

On the other hand, in order to obtain ψ_w the following equation suggested in Cecilio *et al.* (2007) shall be used:

$$\psi_w = \frac{\psi_b (2 + 3\lambda)\left[K_r(\theta_t)^{\frac{3\lambda+1}{3\lambda+2}} - K_r(\theta_0)^{\frac{3\lambda+1}{3\lambda+2}}\right]}{[K_r(\theta_t) - K_r(\theta_0)][3\lambda + 1]} \quad (17)$$

where ψ_b is the air-entry value (mm); λ is the pore size distribution index (nondimensional) and K_r is the relative hydraulic conductivity (nondimensional), expressed as $K(\theta)/K_s$. Based on Brooks & Corey (1964) model, the relative hydraulic conductivity is:

$$K_r(\theta_j) = \left(\frac{\theta_j - \theta_r}{\theta_s - \theta_r}\right)^{\frac{2}{\lambda}+3} \quad (18)$$

where, θ_j is the water content whose relative hydraulic conductivity is sought at; θ_r is the residual water content.

Some propositions related to choosing appropriate parameters which better represent the properties of the soil must be taken into account. Transition zone's water content θ_t can be used instead of the saturation water content θ_s, since θ_t is the water content that most of the soil mass reaches while the infiltration process occurs. Also, the experiments available to obtain the value of K_0 often give a considerable dispersion of the results, what can be minimized by choosing the steady infiltration rate, *Tie*, as representative of the hydraulic conductivity of the soil profile (Cecílio *et al.*, 2007).

4. Examples of the Application of the Proposed Method

The experimental data used in this study are given by Cecílio (2005). Three types of soil will be analyzed, namely: Red Clay Soil (PV), Red Laterite Soil (LV) and Red-Yellow Laterite Soil (LVA). The characteristics of these soils and experimental data obtained (Cecílio, 2005) are presented in Tables 1 and 2 and Figs. 2 and 3.

In order to provide a better characterization of the soils studied, each one is graphically analyzed. It can be verified by experimental data (Cecílio, 2005) that the relative hydraulic conductivity can be graphically related to the volumetric water content as seen on Fig. 2.

It's clear from Fig. 2 that LV soil has a different behavior from LVA and PV soils, which is mainly related to their composition (LV has more sand in its composition while both LVA and PV have more clay), as seen on Table 1.

On the other hand, Fig. 3 shows graphically the relation between the relative hydraulic conductivity and the average wetting front matric potential based on Brooks and Corey model.

Figure 3 also shows that each soil has a unique behavior which mainly depends on the pore size distribution index. This dependence is obtained due to the adoption of Brooks and Corey model for the prediction of the wetting front matric potential.

Based on the data presented by Cecílio (2005) and considering Eq. 11 for predicting the temporal evolution of the cumulative infiltration, the results obtained for the modeling of each soil are presented in Fig. 4.

It can be seen that Eq. 11 provides good predictions of the infiltration behavior. For all the soils analyzed, crescent, however discontinuous, functions have been adjusted

Table 1 - Grain-size distribution of the soils considered.

Soil	Coarse sand (%)	Fine sand (%)	Silt (%)	Clay (%)
LVA	13	10	7	70
LV	26	52	2	20
PV	7	9	25	59

Table 2 - Physical attributes of the soils considered.

Soil	θ_o	θ_s	θ_t	θ_r	ρ (10^3 kgm^{-3})	K_0 (10^{-6} m s^{-1})	Tie (10^{-6} m s^{-1})	i_p (10^{-6} m s^{-1})	ψ_b (10^{-6} m s^{-1})	λ
LVA	0.327	0.541	0.521	0.237	1.05	31.14	36.19	486.0	0.087	0.4032
LV	0.113	0.479	0.398	0.085	1.42	28.50	70.31	530.0	0.132	0.7470
PV	0.300	0.543	0.523	0.237	1.19	9.86	21.92	211.0	0.104	0.3572

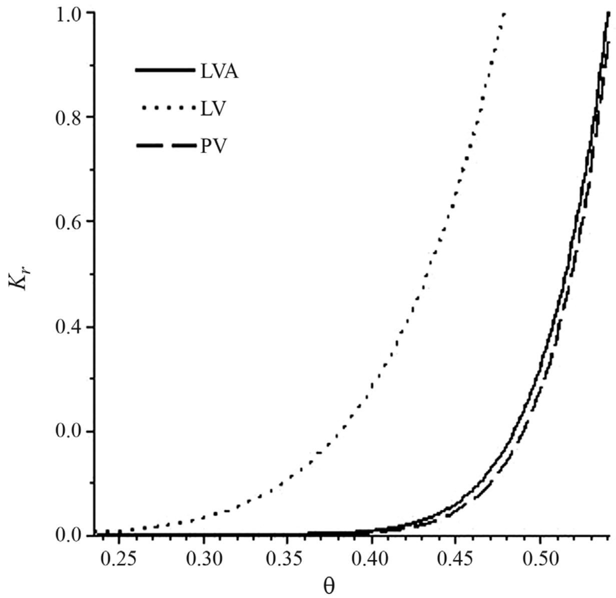

Figure 2 - Relative hydraulic conductivity *vs.* volumetric water content.

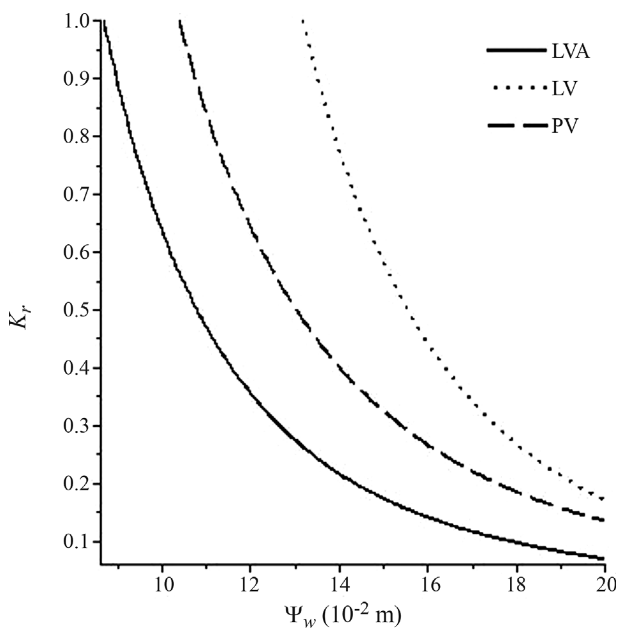

Figure 3 - Relative hydraulic conductivity *vs.* average wetting front matric potential.

for each situation. By choosing α close to 0, the graph tends to overestimate the values of the cumulative infiltration. On the other hand, if such parameter is closer to 1, the values of the cumulative infiltration decrease. It can be seen that the function which predicts the infiltration has a discontinuity when t is equal to t_p. This comes from the assumption that t_p and t_s are obtained by means of the exact three-parameter equation. In order to work out this issue, one shall obtain the explicit equations for each parameter based on the approximated solution proposed. Since such discontinuities do not affect the values of α, this procedure is out of the scope of the present effort. The results are summarized in Table 3.

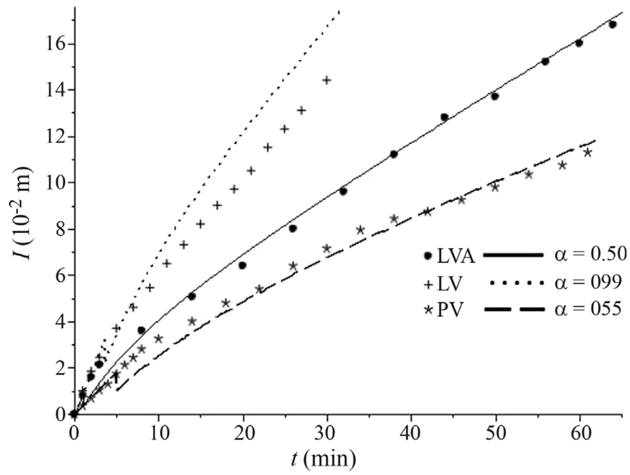

Figure 4 - Cummulative infiltration *vs.* time for the soils analyzed.

Table 3 - Results.

Soil	Classification	$\alpha_{obtained}$
LVA	Red-Yellow Laterite	0.50
LV	Red Laterite	0.99
PV	Red Clay	0.55

5. Conclusions

The three-parameter equation for infiltration prediction has been solved by means of an iterative algorithm whose single-iteration convergence ratio can be chosen as high as desired.

The methodology has been validated by comparing the approximate solutions to the exact ones. The maximum absolute departure of the approximated nondimensional cumulative infiltration is of about 0.23 and happens at low nondimensional times. The results presented considered a single iteration of the algorithm, thus the error could be mitigated by taking more iterations into account.

Based on the three-parameter infiltration equations and the solutions developed, three real cases were studied and the value of the interpolation parameter α has been obtained for each one. For a red clay soil α = 0.55; for a red laterite α = 0.99 and finally, for a red-yellow laterite soil α = 0.50. These values may vary mainly due to the type of soil analyzed.

It’s known that exact solutions are not necessary for practical purposes. The tree-parameter infiltration equation can only be exactly solved in terms of special functions, which ultimately reduces it applicability to practical situations. In the present paper, simple yet powerful approximated solutions with good accuracy are presented. This way, it becomes easier to apply the three-parameter infiltration equation to field studies.

Acknowledgments

The authors acknowledge the support of the following institutions: Brazilian National Research Council (CNPq) and University of Brasilia. The authors would also like to thank two anonymous reviewers for the valuable suggestions on the improvement of the present paper and, finally, Prof. Roberto Cecílio for gently offering the experimental data used in the present effort.

Table of Symbols

a = convergence parameter
I = cumulative infiltration
I_* = nondimensional cumulative infiltration
I_p = cumulative infiltration until ponding time
i_p = rainfall intensity
K_t = hydraulic conductivity in transmission zone
K_r = relative hydraulic conductivity
K_s = saturated hydraulic conductivity
p_s = average surface pressure head
r = infiltration rate
t = time
t_* = nondimensional time
t_p = ponding time
t_s = time necessary to infiltrate the same amount infiltrated until the ponding time, under saturation of the superficial layer of the soil
Tie = steady infiltration rate
α = interpolation parameter;
θ_s = saturated water content
θ_o = initial water content
θ_r = residual water content
λ = pore size distribution index.
ψ_w = average wetting front matric potential
ψ_b = absolute value of the matric potential of air entrance.

References

Brooks, R.H. & Corey, A.T. (1964) Hydraulic properties of porous media. Hydrologic Paper #3, Colorado State University, Fort Collins, 26 p.

Cecilio, R.A. (2005) Modelagem da Infiltração da Água no Solo Fundamentada na Equação de Green-Ampt-Mein-Larson. Tese de Doutorado, Programa de Pós-Graduação em Engenharia Agrícola. Universidade Federal de Viçosa, 164 p (in Portuguese).

Cecilio, R.A.; Martinez, M.A.; Pruski, F.F.; Silva, D.D. & Ataíde, W.F. (2007) Substituição dos parâmetros do modelo de Green-Ampt-Mein-Larson para estimativa da infiltração em alguns solos do Brasil. Rev. Bras. Ciênc. Solo, v. 31:5, p. 1141-1151.

Green, W.H. & Ampt, G.A. (1911) Studies in soil physics. I. The flow of air and water through soils. J Agri. Sci., v. 4:1, p. 1-24.

Householder, A.S. (1970) The Numerical Treatment of a Single Nonlinear Equation. McGraw-Hill, New York, 216 p.

Mein, R.G. & Farrell, D.A. (1974) Determination of wetting front suction in the Green-Ampt equation. Soil Science Society of America Proceedings, v. 38:6, p. 399-417.

Parlange, J.-Y.; Lisle, I.; Braddock, R.D. & Smith, R.E. (1982) The three parameter infiltration equation. Soil Sci, v. 133:6, p. 337-341.

Parlange, J.-Y.; Barry, D.A. & Haverkamp, R. (2002) Explicit infiltration equations and the Lambert W-function. Advances in Water Resources, v. 25:8-12, p. 1119-1124.

Rathie, P.N.; Swamee, P.K.; Cavalcante, A.L.B. & Ozelim, L.C.S.M. (2012) Lagrange's inversion theorem and infiltration. World Academy of Science, Engineering and Technology, v. 67, p. 473-478.

Talsma, T. & Parlange, J.-Y. (1972) One-dimensional vertical infiltration. Australian J. Soil. Res., v. 10:2, p. 143-150.

Zonta, J.H.; Martinez, M.A.; Pruski, F.F.; Silva D.D. & Motebeller, C.A. (2010) Adjustment of Green-Ampt-Mein-Larson model parameters under field conditions. Rev. Bras. Eng. Agric. Ambient., v. 14:10, p. 1021-1029.