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Evaluation of the accuracy of the cubic law for flow through fractures using Lattice Boltzmann method

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Article

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Abstract

In this study, a particle-based approach is employed to simulate the fluid flow through single fractures considering detailed geometrical characteristics of the fracture walls at meso-scale. For this purpose, Lattice Boltzmann Method (LBM) which is an efficient computational fluid dynamic method for simulation of fluid flow in media with grossly irregular geometries is employed. The developed numerical model is validated against some benchmark problems with available theoretical solutions including fluid flow through planar channel with non-parallel walls and non-planar channel with parallel walls. The results indicate the capability of the developed numerical model for simulation of flow through irregular boundary conditions of natural fractures. The effect of the variability of fracture aperture, tortuosity of fracture centerline and roughness of fracture walls on the volumetric flow rate is investigated. Moreover, appropriate definition of hydraulic aperture as the key parameter of the well-known cubic equation for estimation of volumetric flow rate in natural fractures is evaluated by numerical simulation of randomly generated fractures with various geometrical conditions.

1. Introduction

The understanding of fluid flow behavior and accurate estimation of flow rate in fractured media is essential for many engineering applications. Groundwater flow and solute transport in rock mass are typically controlled by fractures as the preferential flow pathways. The estimation of fluid flow through fractured reservoirs for prediction of oil or natural gas production is of great importance in petroleum industry. Moreover, fluid flow behavior in fractured rocks should be appropriately identified in engineering problems such as geothermal energy extraction, liquid waste disposal/injection and grouting activities (Singh et al., 2015).

Laboratory and field measurement are usually considered as reliable sources for obtaining the rate of fluid flow through fractured media. However, most experimental techniques fail to directly observe the details of the flow behaviors in real fractures which are locally influenced by the complex geometry of the fractures (Wang et al., 2016). Alternatively, analytical relationships and numerical modeling can be used for investigation of flow behavior. Because of the inherent flexibility of numerical modeling in incorporating various conditions, these tools can be used besides the laboratory and field tests for fluid flow measurement.

Generally, study of fluid flow in a medium containing a network of natural fractures requires understanding the controlling mechanisms of fluid flow through a single fracture. For precise description of fluid flow through a single fracture, the Navier-Stokes equations of hydrodynamics should be solved. However, fluid flow in a natural fracture, which is normally bounded by two irregular walls with rough surfaces, is complex even under a laminar flow regime. Therefore, fluid flow through fractures is typically conceptualized by using the assumption of laminar flow between parallel plates.

The parallel-plate solution for the Navier-Stokes equations leads to the commonly used "cubic law" (Lomize, 1951; Louis, 1969; Kranzz et al., 1979; Tsang & Witherspoon; 1983; Klimczak et al., 2010, Wang et al., 2015). The cubic law (CL) states that the volume rate of fluid flow across a section in such a fracture is proportional to the applied pressure gradient and the cube of the separation distance. The important implication of the cubic law is that fluid flow may be fully characterized by the separation

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distance, called "aperture," although the velocity varies across that distance (Ge, 1997).

The equivalent aperture that gives the same fluid flow rate by cubic law, called the hydraulic aperture, is normally smaller than the actual opening displacement of the fracture or mechanical aperture (Klimczak et al., 2010). This is mainly due to several important features that are not supported by the simple CL including throats along the path of flow, tortuosity of flow through natural channels and surface roughness of fracture walls (Neuzil & Tracy, 1981; Walsh, 1981; Tsang, 1984; Walsh & Brace, 1984; Tsang & Tsang, 1987; Brown et al., 1995; Nicholl et al., 1999; Klimczak et al., 2010). All of these factors lead to an increase in the energy dissipation over that predicted by the CL, resulting in lower observed flow rates than the ideal.

Several different modifications of the CL have been proposed to account for the abovementioned effects on the flow behavior averaged over a single fracture. These modifications usually have been considered either by applying a correction factor to incorporate other involving factors into conventional cubic law (*e.g.* Lomize, 1951; Louis, 1969; Patir & Cheng, 1978; Witherspoon et al., 1980; Walsh, 1981; Walsh & Brace, 1984; Zimmerman et al., 1991; Gutfraind & Hansen, 1995) or by utilizing alternative definitions of mean aperture (*e.g.* Neuzil & Tracy, 1981; Tsang & Witherspoon, 1981; Brown, 1987; Hakami & Barton, 1990; Tsang & Tsang, 1990; Unger & Mase, 1993; Renshaw, 1995; Nicholl et al., 1999; Méheust & Schmittbuhl, 2001, Baghbanan & Jing, 2007, Akhavan et al., 2012).

In this study, 2D meso-scale modeling of fluid flow through a single fracture using Lattice Boltzmann Method (LBM) has been considered. Unlike the traditional computational fluid dynamics (CFD) methods that directly solve the Navier-Stokes equations, LBM actually simulates macroscopic flows by means of a particulate approach. The particle-like nature of LBM permits a transparent treatment of grossly irregular geometries of fracture walls in terms of elementary mechanical events, such as mirror and bounce-back reflections without comprising great computational costs.

In this work, the applied numerical method has been validated by simulation of the fluid flow through channels with simple geometries. The effects of variable fracture aperture, fracture tortuosity, and roughness of fracture surfaces on the rate of fluid flow through a single fracture has been investigated using the developed numerical tool. Moreover, different definitions of hydraulic aperture have been evaluated by numerical simulation of randomly generated fractures with various geometrical conditions.

2. Cubic law

Solving the Navier-Stokes equations under a set of boundary conditions will provide details on pressure and flow velocity distributions in discrete fractures. In general, the explicit treatment of fracture surfaces as boundary conditions complicates the approach and does not lead to simple solutions. However, according to Zimmerman et al. (1991), under certain geometric and kinematic constraints, the Navier-Stokes equations can be locally reduced to the much simpler Reynolds (Equation 1).

$$\nabla \cdot [d^3(x,y)\nabla p] = 0 \tag{1}$$

where (x, y) are orthogonal coordinates in the plane of the fracture, d is local aperture of the fracture, and p is fluid pressure. One of the requirements for the Reynolds equation to be valid is the viscous forces dominate the inertial forces (very small Reynolds number). Equation 1 is a linear partial equation that describes the pressure field in the fracture plane. The volumetric flow rate per unit width perpendicular to the direction of flow is then related to the pressure, as presented in Equation 2 (Zimmerman et al., 1991).

$$Q = \frac{d^3(x,y)}{12\mu}\nabla p \tag{2}$$

where Q is the volumetric flow rate, and $\mu$is the dynamic viscosity of the fluid.

For an ideal laminar flow through two smooth parallel plates, Equation 2 is simplified to the well-known cubic law (CL), which has the form described by Equation 3 (Bear et al., 1993).

$$Q = \frac{d^3}{12\mu}\frac{\Delta p}{\Delta L} \tag{3}$$

where $\Delta p/\Delta L$ is the magnitude of the pressure gradient.

Due to the approximate planar nature of fractures, flow behavior in discrete fractures has often been assumed to be similar to flow through two smooth, parallel plates. The major factor causing deviation of predicted fracture flow behavior from the ideal parallel plate theory is the nature of non-parallel and non-smooth geometry of fracture surfaces. Important questions on the validity of the cubic law and the Reynolds equation for complicated fracture geometries have been studied by many researchers (Witherspoon et al., 1980; Tsang & Witherspoon, 1983; Walsh, 1981; Brown, 1987; Zimmerman et al., 1991; Bear et al., 1993; Oron & Berkowitz, 1998; Brush & Thomson, 2003). The general conclusion from these efforts is that the cubic law is valid provided that an appropriate mean aperture (hydraulic aperture) is used and modification factors for tortuosity and surface roughness is applied. Many types of mean apertures have been proposed, and for some cases, some averaging procedures work better than others (Ge, 1997). Some of the proposed hydraulic apertures will be discussed in this paper.

It should also be noted that laminar flow is assumed in derivation of cubic law and therefore, the inertial effects of flow is ignored where cubic law is used. For flows with high Reynolds numbers (*Re*), large fluid velocity gradient across the aperture will occur. This results in a broader distribution of the immobile regions along the rough fracture wall (Dou & Zhou, 2014). The immobile region is enlarged with the increase of *Re* and roughness, and its impact cantnot be captured using conventional cubic law. To characterize the nonlinear flow through fractures, the complete form of Navier-Stokes equations including the acceleration and inertial terms should be adopted as the governing equation (Wang et al., 2016).

Surface roughness is one of the most important factors affecting the hydro-mechanical behaviour of rock fractures. Pioneer study by Lomize (1951) was one of the comprehensive efforts to indicate the accuracy of the cubic law for laminar flow under low hydraulic gradient by considering a correction factor for surface roughness. Later, the equation was confirmed by some other researchers (*e.g.* Snow, 1965). Based on extensive experimental studies, Louis (1969) proposed a similar reduction factor to be applied to the cube law (Equation 4).

$$Q = \frac{d^3}{12\mu}\frac{\Delta p}{\Delta L}\frac{1}{F}; \quad F = (1 + 8.8R_r^{1.5}) \tag{4}$$

where R_r is relative roughness factor.

In this approach, the friction factor depends only on the relative roughness and therefore, ignores the frequency (or wavelength) of the asperities (Zambrano et al., 2019). The roughness correction factor can be also calculated based on the joint roughness coefficient (JRC) as proposed by Barton et al. (1985). JRC has been widely used in geotechnical and rock engineering applications and several researchers used this coefficient to study the fluid flow through rock fractures (Zhang & Nemcik, 2013; Crandall et al., 2010; Niya & Selvadurai, 2019). However, the JRC is estimated subjectively based on visual inspection of the roughness profiles and the results always vary based on the experience level of the investigator (Su et al., 2020). Moreover, the accuracy of JRC for relatively wide apertures decreases due to its moderate resolution, which is about 1 mm (Zambrano et al., 2019).

Tortuosity is another important factor that influences the connectivity of fluid flow paths because of fracture contact area (Tsang 1984). Walsh & Brace (1984) investigated the effects of tortuosity and presented an equation for flow through the fractures (Equation 5).

$$Q = \frac{d^3}{12\mu}\frac{\Delta p}{L}\frac{1}{\tau^2} \tag{5}$$

In Equation 5, the parameter τ is the curvature coefficient of tortuosity that equals to the ratio of the actual length to the apparent length of the flow path. The same tortuosity modification factor was used by Nazridoust et al. (2006).

On the other hand, some researches focused on finding modification coefficient based on distribution function characteristics of the aperture (Tsang & Tsang, 1990; Renshaw, 1995; Zimmerman & Bodvarsson, 1996). For example, Equation 6 presented by Zimmerman & Bodvarsson (1996) allows the estimation of the volumetric flow rate through rough fractures.

$$\frac{Q}{B} = \frac{d^3}{12\mu}\frac{\Delta p}{L}C_r C_t \tag{6}$$

where C_r and C_t are roughness and tortuosity correction factor, respectively that can be calculated based on arithmetic average and standard deviation of aperture variation.

Various definitions of mean aperture (d), including the arithmetic mean (AM), geometric mean (GM), harmonic mean (HM), volume averaged mean (VAM), and effective aperture (EA) have been proposed to be applied into the conventional cubic law in order to account for important features that are not supported by the simple cubic law (Akhavan et al., 2012). Most studies show that for natural fractures, the cubic law calculates the amount of the flow more than its actual quantity, especially if the arithmetic mean value of aperture is used in the cubic law (Konzuk & Kueper, 2004). In some studies, geometric average has been suggested as a better representative of the hydraulic aperture (Jensen, 1991; Renshaw, 1995; Konzuk & Kueper, 2004; Baghbanan & Jing, 2007). Some other researchers believe that the flow is controlled by the least aperture width along the flow path (*e.g.* Pyrak-Nolte et al., 1988).

Nazridoust et al. (2006) proposed an effective fracture aperture (H_{eff}) based the average aperture (H_{avg}), and the standard deviation of the fracture apertures (σ). For a normal distribution of the apertures, H_{eff} given by Equation 7 is an estimate of smaller apertures (the probability of a randomly selected aperture in the fracture that is larger than H is 84.14 %).

$$H_{eff} = H_{avg} - \sigma \tag{7}$$

By extending a technique originally suggested by Dietrich et al. (2005), Akhavan et al. (2012) indicate that effective aperture as defined by the following equation is the appropriate representative of hydraulic aperture to be applied in cubic law:

$$d_{eff} = \sqrt[3]{\frac{1}{n}\sum_{j=1}^{n} d_j^3} \tag{8}$$

The equation can be derived simply by discretization of a fracture into a series of local parallel channels with different apertures (d_j), and n is the number of fracture elements.

As an alternative approach to conventional CL, Equation 1 can be used to consider the local variation of aperture explicitly. The Local Cubic Law (LCL) has been extensively applied in investigations of fluid flow through fractures (Zimmerman et al., 1991; Mourzenko et al., 1995; Nicholl et al., 1999; Wang et al., 2015). Under LCL, local flow magnitude is proportional to the cube of the local aperture. The local aperture can be measured using different methods (Mourzenko et al., 1995; Ge 1997; Oron & Berkowitz 1998; Wang et al., 2015). Ge (1997) provided analytical solutions for fluid flow through two-dimensional non-parallel and parallel fractures by considering the normal-to-local-centerline aperture. Wang et al. (2015) modified the LCL to take into account the local tortuosity and roughness, and low inertial effects where local Re ≤ 1. The proposed LCL is more accurate than previous modifications of the LCL. However, the governing equation is a nonlinear differential equation that should be solved numerically.

The focus of this study is to investigate the use of hydraulic aperture in conventional cubic law. For this purpose, several hydraulic aperture definitions have been numerically investigated using the developed meso-scale model. The numerical model has been validated against analytical solutions proposed by Ge (1997) using local cubic law approach for two problems including fluid flow through single fracture with regular geometries.

3. Lattice Boltzmann method

During the last two decades, particle-based methods such as Lattice Boltzmann Method (LBM) have been developed as a robust numerical approach in computational fluid dynamics (CFD). In this method, macroscopic flow is simulated by means of a particulate approach. It can be considered as a special finite-difference form of the continuum Boltzmann equation, but historically it is a pre-averaged improvement to its predecessor, the lattice-gas method (Sangani & Acrivos, 1982; Chen & Doolen, 1998; Succi, 2001).

The fundamental idea of the LBM is to construct simplified kinetic models that incorporate the essential physics of microscopic or mesoscopic processes, so that the macroscopic averaged properties obey the desired macroscopic equations (Pan et al., 2001). Unlike most of the other particle methods, LBM is a mesh-based method. In LBM, the spatial space is discretized in a way that it is consistent with the kinetic equation (Pan et al., 2001). The LBM simulates the flow phenomenon by tracking fluid particles that move and collide in space under the rules that the collision does not result in mass and momentum changes.

In LBM, space is divided into regular lattices normally with the same spacing h in both directions and at each lattice site a particle distribution function $f_i(x, t)$ is defined which is equal to the probable amount of fluid particles at site x moving in the direction of i at time t. During each discrete time step of the simulation (Δt), fluid particles move to the nearest lattice site along their direction of motion with different velocities of $\vec{e}_i$, where they "collide" with other bundles of fluid particles that arrive at the same site. The outcome of the collision is determined by solving the kinetic (Boltzmann) equation for the new particle distribution function at that site and the particle distribution function is updated. The magnitude of speed in different directions which is called lattice speed is defined as $C = h/\Delta t$.

Propagation and collision of fluid particles in LBM can be mathematically summarized by the below two-step scheme (Eqs. 9 and 10):

Propagation step:

$$f_i(x+\vec{e}_i\Delta t, t+\Delta t) = f_i^{updated}(x,t) \tag{9}$$

and collision step:

$$f_i^{updated}(x,t) = \Omega_i(f(x,t)) \tag{10}$$

The collision rule Ω should be chosen to leave the sum of the $f_i(x, t)$ unchanged (no fluid particles are lost.) The rule is also selected to conserve the total momentum at each lattice site. The collision process is mimicked by a distribution function rather than solving for collisions of every fluid particle. Lattice-Boltzmann models can be constructed using Fermi-Dirac or Maxwellian distributions as the collision process (Engler, 2003). However, solving these distribution functions is complicated and computationally expensive.

The single relaxation time operator, also known as Bhatnagar-Gross-Krook (BGK) operator after Bhatnagar et al. (1954), is an uncomplicated approach which simply approximates the collision by assuming that the momentum of the interacting fluid particles will be redistributed at some constant rate toward an equilibrium particle distribution function f_i^{eq}. BGK allows one to solve the equilibrium distribution such that the microscopic equations are satisfied and the N-S equations are recovered. In BGK lattice Boltzmann method, the collision rule is given by:

$$\Omega_i(x,t) = f_i(x,t) - \frac{\Delta t}{\lambda}(f_i(x,t)) - f_i^{eq}(x,t) \tag{11}$$

where f_i^{eq} is the local equilibrium density distribution for the fluid and λ is relaxation parameter.

The two-dimensional model implemented in this study uses a square, nine-velocity lattice typically referred to as D2Q9 model. For this model, f_i^{eq} is given by Eqs. 12 and 13, presented by Qian et al. (1992).

$$f_0^{eq} = w_0\rho\left(1-\frac{3}{2C^2}\vec{V}\cdot\vec{V}\right) \tag{12}$$

$$f_i^{eq} = w_i\rho\left[1+\frac{3}{C^2}(\vec{e}_i\cdot\vec{V})+\frac{9}{2C^2}(\vec{e}_i\cdot\vec{V})^2 - \frac{3}{2C^2}(\vec{V}\cdot\vec{V})\right], \quad i=1,2,\ldots,8 \tag{13}$$

where ρ and V are macroscopic fluid density and velocity and w_i are the fixed weighting values $w_0 = \frac{4}{9}$, $w_{1,2,3,4} = \frac{1}{9}$ and $w_{5,6,7,8} = \frac{1}{36}$.

The macroscopic parameters are regained by:

$$\rho = \sum_{i=0}^{i=8} f_i \tag{14}$$

$$\rho \cdot \vec{V} = \sum_{i=0}^{i=8} f_i \cdot \vec{e}_i \tag{15}$$

$$P = C_s^2 \rho \tag{16}$$

where P is pressure and C_s is termed fluid's speed of sound which is related to the lattice speed by $C_s = C/\sqrt{3}$. The dimensionless relaxation parameter $\tau = \lambda/\Delta t$ is related to the kinematic viscosity of the fluid ν by Equation 17.

$$\nu = \frac{1}{3}\left(\tau - \frac{1}{2}\right)\frac{h^2}{\Delta t} \tag{17}$$

A constraint to the parameter selection is that the lattice speed C must be sufficiently larger than the maximum fluid velocity (V_{max}) in the simulation to ensure a sufficient solution accuracy. This is calculated by the 'computational' Mach number, defined by $Ma = V_{max}/C$. Theoretically it is required that $Ma \ll 1$. In practice, Ma should be, at least, smaller than 0.1 (Feng et al., 2007). This becomes very important in modeling fluid flow through packing of solid particles, when fluid particles may have high velocities in small communication between channels (Succi, 2001).

In classical fluid dynamics, the interface between the solid boundaries and the flowing fluid is assumed to be a non-slip boundary. Simulating slip and non-slip boundaries in the LBM is an area where progress is still being made (Qian et al., 1992; Noble et al., 1995). Among the existing methods, the simplest is called 'bounce-back' method. In this approach, to ensure that the fluid particles have zero average velocity at the boundaries (both perpendicular and parallel to the walls): any flux of fluid particles that hits a boundary simply reverses its velocity so that the average velocity at the boundary is automatically zero. This type of boundary is utilized in this study. For this purpose, fracture walls should be characterized by the lattice nodes. The discrete nature of the lattice will result in a stepwise representation of the curve boundaries, so for acquiring the required accuracy and smoothness, sufficiently small lattice spacing is used.

4. Model validation

4.1 Planar channel with non-parallel walls

Equation 3 is valid for a smooth, straight, and parallel plate fracture. However, natural fractures rarely have such characteristics. Here, to validate the LB model in simulation of fractures with irregular walls, fluid flow through a planar channel but with zigzag walls, as illustrated in Figure 1a, was modeled. Higher ratios of zigzag height to channel width (h/d_m) show channels with variable width along the length of the channel. On the other hand, lower h/d_m. ratios represent channels with rough wall surfaces. Ge (1997) provided an analytical solution for this problem using a general governing equation based on the principle of mass conservation and the assumption that CL is locally established. In this solution, the fracture walls should be described by an explicit function $f(x)$. For the fracture illustrated in Figure 1a, the top surface of the fracture can be described mathematically using Fourier series (Equation 18).

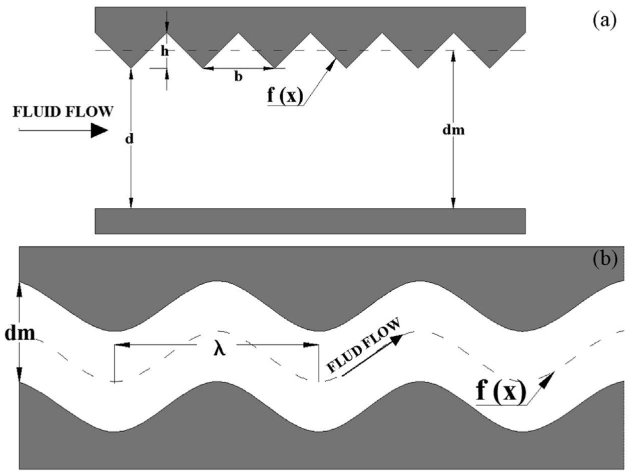

Figure 1. a) planar channel with non-parallel walls, b) non-planar channel with parallel walls.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2n\pi}{b}x\right) \tag{18}$$

where a_0 and a_n are Fourier coefficients and can be found from the following equation.

$$a_n = \frac{h}{(n\pi)^2}(2\cos n\pi - \cos 2n\pi - 1)$$
$$\text{and} \quad a_0 = \frac{2}{b}\int_0^b f(x)dx \tag{19}$$

where b and h are geometrical characteristic of the top fracture wall as defined in Figure 1a.

All parameters are shown on Figure 1a. Based on Ge (1997)'s solution, true aperture at a given x in the direction of the flow can be defined as follows:

$$m(x) = C(x)d_a = C(x)f(x) \tag{20}$$

$$C(x) = \frac{2\cos(\alpha(x))}{\cos(\frac{(\alpha(x)}{2})(1+\cos((\alpha(x)))} \tag{21}$$

$$\alpha(x) = \tan^{-1}(f'(x)) \tag{22}$$

where $\alpha(x)$ is the slope of the top surface, d_a is apparent aperture (distance between top and bottom surfaces for a given x), $m(x)$ is true aperture for a given x and f' is derivative of f. Zimmerman et al. (1991) show that the Reynolds equation can lead to the following expression for the hydraulic aperture:

$$d_h^3 = \frac{1}{x_2 - x_1}\left[\int_{x_1}^{x_2} \frac{dx}{m(x)^3}\right]^{-1} \quad (23)$$

The above equation gives the appropriate hydraulic aperture for estimating flow rate by the cubic law. In this study, LBM was applied for simulation of the flow in the same problem. For this purpose, a domain including a 300 × 40 lattices with spacing h = 0.0002 cm and dimensionless relaxation parameter τ = 0.65 was considered. Pressure boundaries were applied at both downstream and upstream sides of the channel and no-slip boundary condition was assumed at the left and right of the domain to model the rigid walls.

The results of numerical simulations as well as flow rates obtained by CL by application of hydraulic aperture (d_h) are presented in Figure 2a. The numerical results are in good agreement with the analytical solution for different sizes of surface zigzags. Both results show that the volumetric flow rates increase with the decreases of h/d_m. Also, as shown in Figure 2b, where the size of zigzags on top wall is the same for all analyses, both analytical solution and LB results imply that flow rates do not follow the CL in which the flow rate is a function of the cube of the average channel

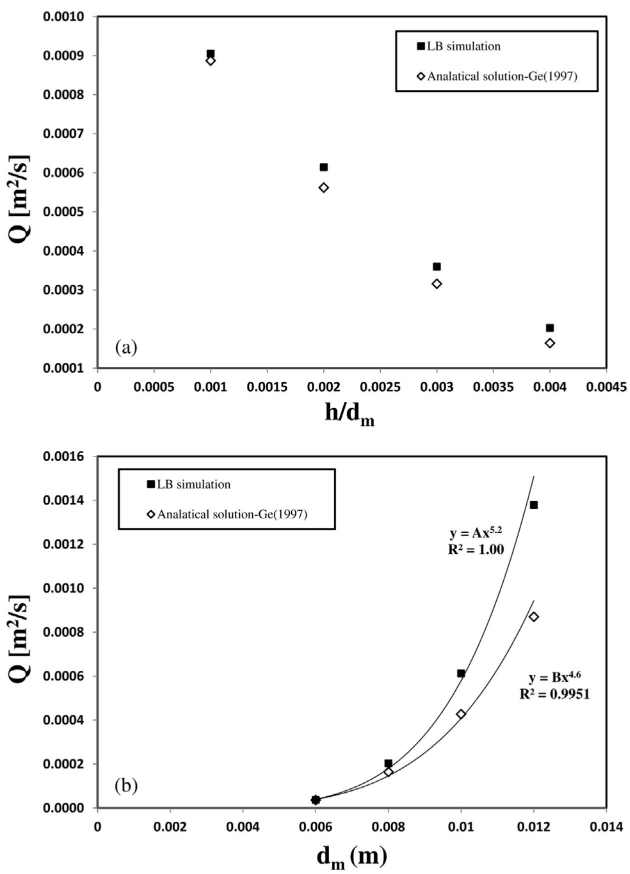

Figure 2. Comparison between flow rate obtained by numerical results and Ge's (1997) solution: a) different sizes of surface zigzags, b) different mean aperture.

width d_m. This deviation confirms the inaccuracy of using average aperture for estimation of flow rate by conventional CL equation.

4.2. Non-planar channel with parallel walls

Another benchmark problem considered to validate the developed LB model is simulation of flow through channels with non-planar walls as shown in Figure 1b. The mathematical equation for centerline of this channel is given by a sinusoidal function as follows:

$$f(x) = \delta \sin\frac{2\pi x}{\lambda} \quad (24)$$

where δ and λ are amplitude and wavelength of the channel centerline, respectively.

Because the Reynolds equation does not account for tortuosity, the viscous force due to tortuosity is totally ignored. Consequently, this oversight can cause errors in fracture permeability estimation and in the accuracy of the cubic law. In an effort to examine the cubic law under a two-dimensional fracture condition, Brown (1987) used tortuosity factor to correct the calculated flow rate. For this problem, the tortuosity can as shown in Equation 25.

$$\tau(x) = \left[1 + \left(\frac{2\pi\delta}{\lambda}\right)^2 \cos^2\frac{2\pi x}{\lambda}\right]^{1/2} \quad (25)$$

Based on Equation 25, Ge (1997) obtained the following expression for hydraulic aperture for estimation of volumetric flow rate by the cubic law (Equation 26).

$$d_h^3 = \frac{1}{\lambda}\left[\int_0^{\lambda} \frac{dx}{m(x)^3}\right]^{-1} \quad (26)$$

Here, analyses with different characteristics of channel centerline equation are considered. As shown in Figure 3a, the results show that the increase of the wavelength, while the amplitude of the centerline curve remains constant, increases the flow rate. The rate of increase is considerable for lower values of wave lengths. The sensitivity of tortuosity factor to the variation of wavelength can be obtained with Equation 25. In Figure 3a, comparison is also made between LB results and analytical solution proposed by Ge (1997) that shows relatively good agreement especially for lower values of wavelengths.

Figure 3b presents the variation of flow rate with amplitude of channel centerline for a case with constant wavelength. As the amplitude of the channel centerline increases, the ratio of the length of the actual path of flow to the shortest path length in the direction of the flow (tortuosity factor) increases that in turn decreases the rate of the flow.

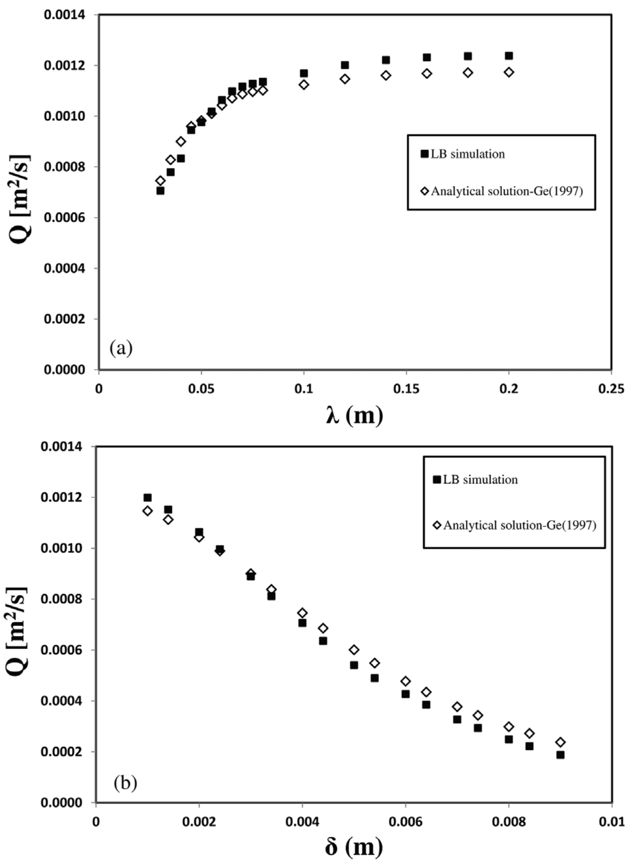

Figure 3. Comparison between flow rate obtained by numerical results and Ge's (1997) solution: a) different wavelengths, b) different amplitudes.

5. Simulation of flow through natural fractures

5.1 Fracture tortuosity

One of the main differences between natural fractures and straight channels is tortuosity. Common definition of tortuosity is the ratio of the length of the fracture centerline to the shortest path length in the direction of the flow. Therefore, tortuosity is equal to unity for straight channels and has larger values for natural fractures. Also, other definitions have also been proposed for tortuosity factor such as those by Zhang & Nagy (2004) and Ghassemi & Pak (2011). However, application of such definitions is mostly limited to researches. Herein, the traditional definition of tortuosity is applied for evaluation of fracture tortuosity.

The results of numerical analyses conducted by changing amplitude and wavelength of a sinusoidal fracture while other parameters were constant are shown in Figure 4. As shown in the figure, for all simulations volumetric flow rate is inversely proportional with tortuosity factor. It should be noted that the tortuosity of flow in natural 3D fractures is affected by the complexity of the velocity field induced by factors such as small and/or large scale roughness, out-of-plane tortuous flow paths and the change of flow direction (Cook, 1992, Oron & Berkowitz, 1998; Wang et al., 2015). These factors are not fully captured by the simple conventional definition of tortuosity. For example, conventional definition of tortuosity only accounts for the curvature of the fracture centerline, but it is insensitive to the form of curvature which governs the change of flow path direction. As shown in Figure 5, the curve of the largest half-circle has the same length as the curve composed of a series of smaller half-circles. However, the results of numerical simulation of flow through channels with equal length but different number of half-circles indicate that the maximum effect of number of half-circles on the flow rate is less than 5 %.

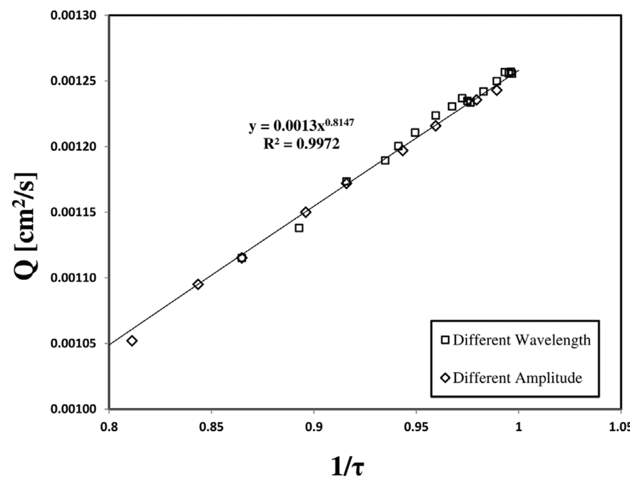

Figure 4. Variation of flow rate with tortuosity factor.

5.2 Throats and surface roughness

Even through a fracture with straight centerline where tortuosity factor is unity, abrupt constrictions of channel walls as well as surface roughness can significantly affect the rate of flow. Herein, a series of analyses in channels with different height (h), length (b) and number of throats was considered (see Figure 6). Furthermore, surface roughness was also investigated by modeling a channel including

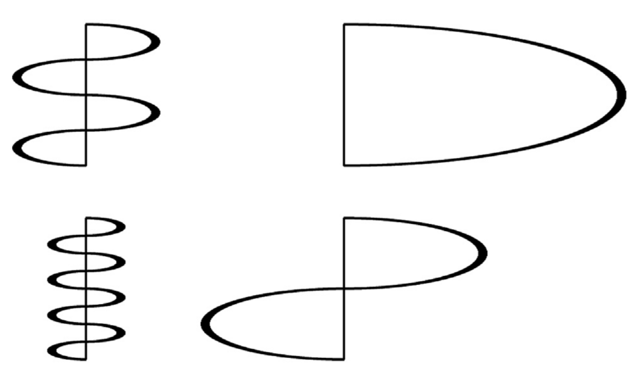

Figure 5. Curves with the same length but different number of half-circles.

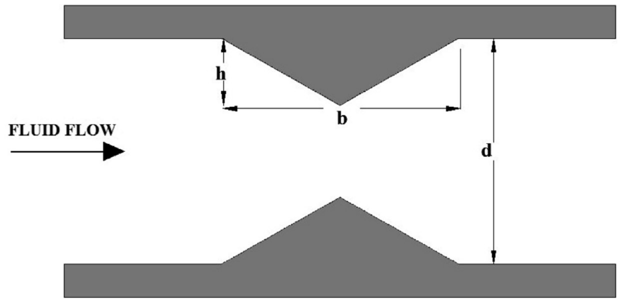

Figure 6. Throat in a channel with straight centerline.

regular subsequent constrictions with low values of h/d_m. As mentioned in Section 2, a couple of relationships for hydraulic aperture have been proposed to be applied in the well-known cubic law for more accurate estimation of flow rate through single fracture. Some of these definitions were examined by the results of LB numerical simulations as shown in Figure 7.

Figures 7a and 7b indicate the variation of the volumetric flow rate over arithmetic and geometrical mean of the channel width along the centerline of the fracture. As can be seen in these figures, although applying geometrical mean of fracture width decrease the scatter of the results, for both methods distinct linear trends are observed for each group of analyses. It should be noted that the appropriateness of meaning method can be related to the form of fracture centerline. For example, use of the geometric mean for isotropically correlated apertures with a lognormal distribution will result in obtaining reasonably flow rates using cubic law (Konzuk & Kueper, 2004).

Similar pattern was obtained by using the correction factor (Cr) proposed by Zimmerman & Bodvarsson (1996) as shown in Figure 7c. However, a relatively well-fitted linear trend can be found in Figure 8a where for all numerical simulations, the results are presented in the form of effective width as defined by Akhavan et al. (2012).

Among the results presented in this figure, some data were extracted from analyses of flow through channels without any variation in width but with rough walls (by considering several constrictions with low values of h/d_m). From Figure 8a, it can be found that the slope of linear trend for these set of data (shown with non-filled symbols) is fairly different from other results shown in the figure. In this study, in order to enhance the convergence of the results to the linear trend, some of different reduction factors proposed in the literature to be applied to the cube law to account for the effects of wall roughness were examined. The outcome implies that modification factor, as defined in Equation 6, gives the best correlation for the conducted analyses. Comparison of Figures 8a and 8b confirms that higher R-squared value can be reached if reduction factor proposed by Louis (1969) is applied to the cube of effective width.

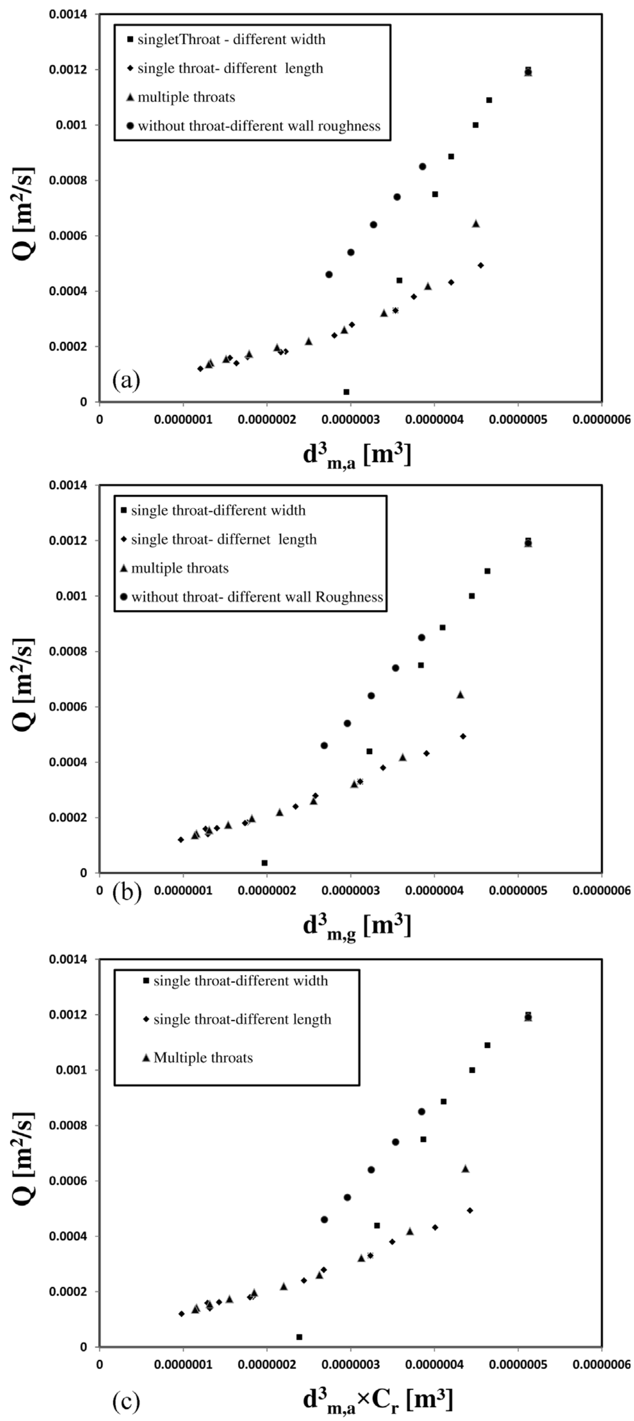

Figure 7. Variation of the volumetric flow rate over mean channel width a) arithmetic average, b) geometrical average, c) modified arithmetic average.

5.3 Randomly generated fractures

Single fractures in geo-materials such as rock mass or concrete always contain tortuous flow paths with variable aperture and rough wall surfaces. In previous parts of this paper, different important geometrical parameters influencing flow through channels such as tortuosity factor, average width and wall roughness were investigated by the developed LB numerical model independently. Herein, attempt is made to study the effects of these parameters on flow through fractures with more realistic geometrical characteristics.

There are various methods to construct a synthetic fracture with rough wall surfaces such as the successive random addition method, the randomization of the Weierstrass function based on Mandelbrot, and the Fourier transformation (Dou & Zhou, 2014). In this study, a random generation process similar the approach used by Yang (2014) was applied to provide fractures with irregular geometries to be used in the fluid flow analyses. In this procedure, the shape of the centerline is defined as a backbone for the fracture. Also, it is assumed that the fracture walls have a zigzag shape and the total numbers of zigzags are defined.

A certain number of random numbers are generated and they are assigned as slopes for those zigzag lines. A scale factor is used to adjust the slopes, if necessary. Another set of random numbers were generated (between 0.5 and 1) for fracture widths. These random numbers are multiplied with appropriate width factor to get the desired fracture width at a given point on the zigzag line. Hence numbers of zigzag lines, width factor and scale factor are the parameters that can be controlled in the process of generation of the fracture. The mentioned procedure was applied to a MATLAB code to generate fractures with random wall geometries which are the boundary conditions for the fluid flow analyses by LB model. A representative illustration of some of the generated fractures by the mentioned procedure is shown in Figure 9.

Several flow simulations through generated fractures with different geometrical properties are conducted to examine the validity of cubic law for estimation of the flow rate using different definition of hydraulic aperture. For comparison, the results of flow rate for these analyses are presented in terms of arithmetic mean, geometrical mean and effective width in Figures 10a, b and c, respectively. These figures indicate that application of the effective width in the cubic law can significantly improve the R-squared value of the obtained data. Therefore, the effective width can be evaluated as an appropriate hydraulic aperture of the fractures for estimation of flow rate using global cubic law.

6. Conclusion

In this study, Lattice Boltzmann Method (LBM) was employed for simulation of fluid flow through channels with grossly irregular wall geometries representing the condition of natural fractures in the geo-materials such as rock and concrete. The developed numerical model was validated against analytical solutions proposed by Ge (1997) using local cubic law approach for two fluid flow

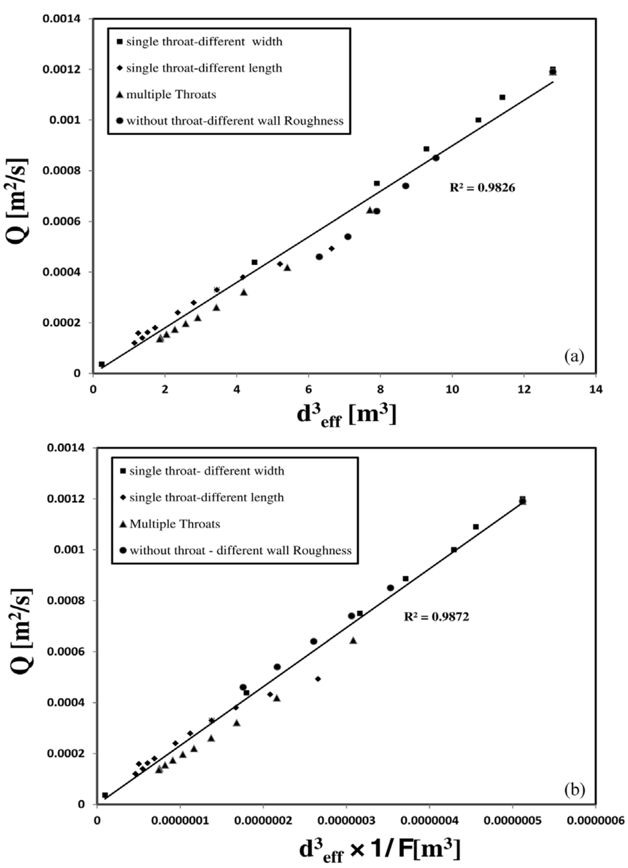

Figure 8. Variation of the volumetric flow rate over a) effective width as defined by Akhavan et al. (2012), b) modified effective width using the reduction factor by Louis (1969).

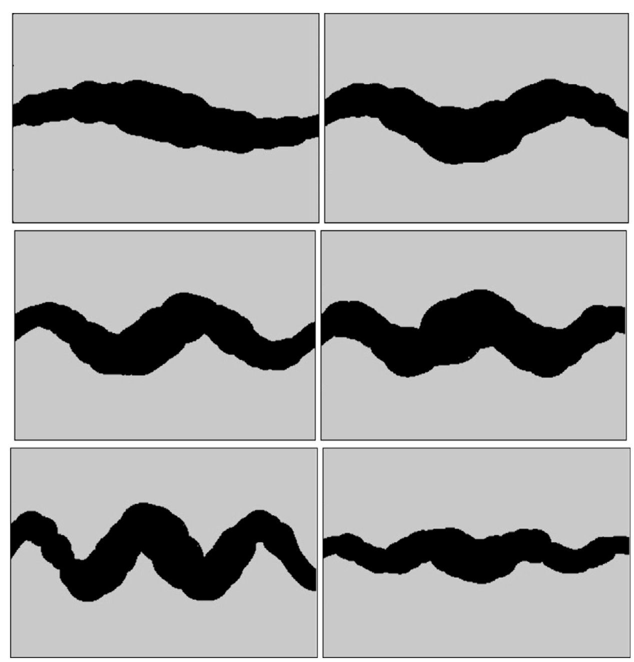

Figure 9. A representative illustration of some of the randomly generated fractures.

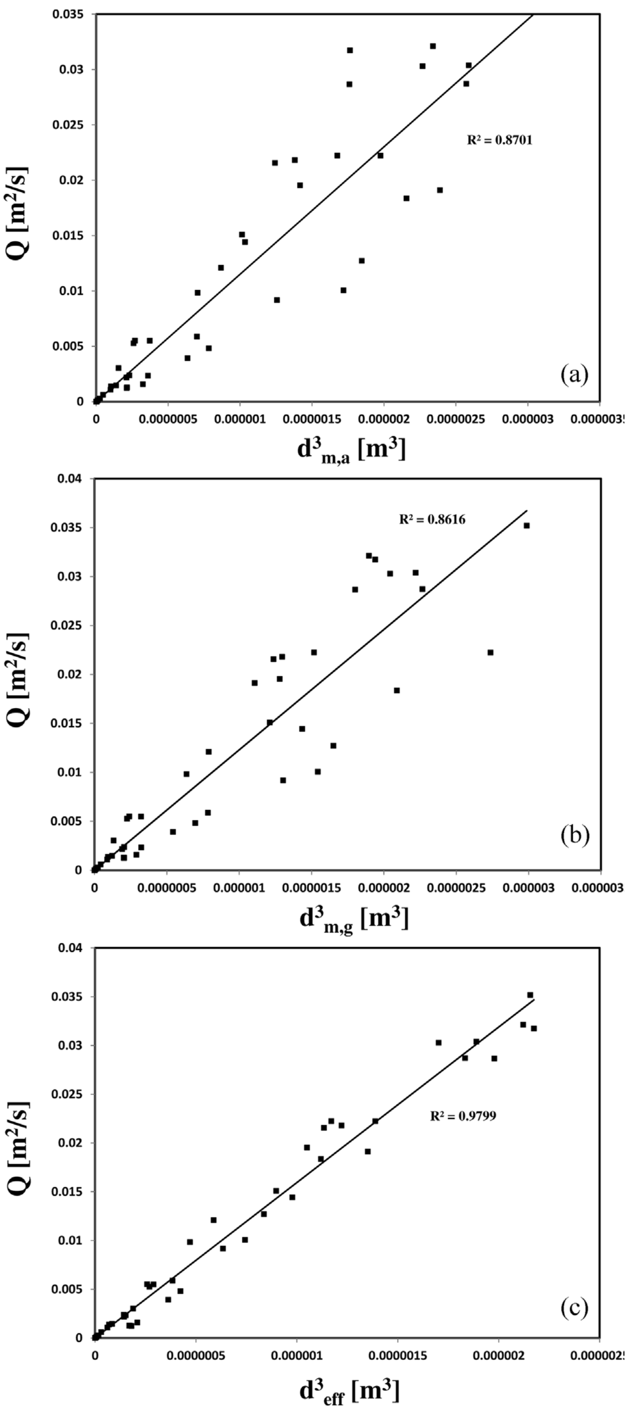

Figure 10. Variation of the volumetric flow rate over a) arithmetic average of width, b) geometrical average of width, c) effective width for randomly generated channels.

problems through single fracture with regular geometries. In general, the obtained results indicate that in spite of the fact that certain physical features of real fractures cannot be fully represented by the applied 2D model; the LBM is a promising numerical method that can be used for estimation of fluid flow through natural fractures.

The focus of the study was evaluation of different definition of hydraulic aperture for implication in the conventional cubic law implying that the volume rate of fluid flow across a section in a fracture is proportional to the cube of the hydraulic aperture.

Based on the obtained results, the following conclusions can be drawn:

1) Volumetric flow rate is inversely proportional with tortuosity factor. Furthermore, the numerical results indicate that flow rate is sensitive not only to the length of centerline curve, but depends on its curvature variation. However, the error of mere consideration of distance ratio in the conventional definition of tortuosity factor is not significant.
2) Even through a fracture with straight centerline (where tortuosity factor is unity), abrupt constrictions of fracture can considerably affect the rate of flow. Numerical results indicate that applying effective width instead of arithmetic or geometrical average of channel width can significantly improve the linear trend expected by the cubic law.
3) The numerical results imply that stronger correlation between the flow rate and the cube of effective width can be reached if reduction factor proposed by Louis (1969) is applied to the cube of effective width for channels with rough surfaces.
4) Flow simulations through randomly generated fractures with different geometrical properties imply that application of the effective width can be evaluated as a more appropriate average parameter representing the hydraulic aperture of the fractures for estimation of flow rate by the conventional cubic law.

More investigations are required for overcoming the limitations of the developed numerical tool such as simulation of non-Darcy flow and implication of 3D formulation in the model. The future work of this research program is the study of the fluid flow in natural fracture profiles using three-dimensional model.

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