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A simple approach to predict settlement due to constant rate loading in clays

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Technical Note

Keywords
Consolidation
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Soft clays
Linearly increasing load

Abstract
Classical theory of consolidation was conceived considering loads instantaneously applied. Since then, researchers have addressed this issue by suggesting graphical and/or analytical solutions to incorporate different time-depending load schemes. The simplest alternative is to assume a linearly increasing load. Another approach to predict the average degree of consolidation caused by a constant rate loading is based on instantaneous excess pore pressures during and at the end of construction. This technical note explains why and how this approach leads to substantial errors after the end of construction. A corrected solution is then proposed, based on the concept of superposition of effects. The final set of equations agree with the theoretical ones. A new simple approximate methodology is also presented. Numerical examples using the proposed approach showed an excellent agreement with the analytical solution. The validity of this new approach was also proven by reproducing oedometer test results with a good agreement.

1. Introduction

Consolidation is one of the most relevant and most studied phenomena in Geotechnical Engineering. The process involves volume change due to water flow in response to stress increase and it is particularly relevant with saturated clayey soils. Due to the extremely low permeability of clays, consolidation can last for decades.

Terzaghi and Fröhlich's (1936) one-dimensional consolidation theory is based on Darcy's law and uses a set of simplifying hypotheses. One of the main assumptions imposes that load is applied instantaneously. The average degree of consolidation U is given by

$$U(T) = 1 - \sum_{m=0}^{\infty} \frac{2}{M^2} e^{-M^2 T} \tag{1}$$

where

$$M = (2m+1)\pi / 2 \tag{2}$$

for m = 1, 2, 3… and the Time Factor T is defined as

$$T = c_v t / H_d^{\,2} \tag{3}$$

Where t is time, H_d is the maximum length of the drainage path and c_v is the coefficient of consolidation.

The consideration of instantaneous load is unlikely to occur in engineering practice. Loading is generally carried out gradually, in stages, during a given construction period. Therefore, consolidation takes place while loading is still in progress.

Several methods have addressed the time-dependent loading issue (Terzaghi, 1943; Schiffman, 1958; Schiffman and Stein, 1970; Olson, 1977; Zhu and Yin, 1998; Conte and Troncone, 2006; Liu and Ma, 2011, Hanna et al. 2013; Gerscovich et al., 2018). The two most worldwide known approaches for linearly increasing loading are Terzaghi's (1943) graphical method and Olson's (1977) theoretical solution.

Terzaghi's (1943) empirical method is a graphical procedure performed differently before and after construction. After the end of construction, the settlement curve is shifted by half the construction period t_c. During construction, the calculation considers only a fraction of the total load as if applied instantaneously in half the time. Despite being a graphical method, whose accuracy is subject to the operator expertise, it can be expressed by the following set of equations (Gerscovich et al., 2018):

$$U'(T) = \begin{cases} \dfrac{T}{T_c} U\left(\dfrac{T}{2}\right) \ldots\ldots\ldots\ldots T \le T_c \\ U\left(T - \dfrac{T_c}{2}\right) \ldots\ldots\ldots\ldots T \ge T_c \end{cases} \tag{4}$$

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Where $U'(T)$ is the corrected average degree of consolidation, $U(T)$ is the corresponding value for instantaneous loading (Equation 1) and T_c is the Time Factor corresponding to the end of construction t_c.

Olson (1977) developed a mathematical solution for a linearly increasing loading as an extension of Terzaghi and Fröhlich's (1936) theory. The ramp load was discretized into small instantaneous incremental loads. Using the principle of superposition, Olson (1977) could express the average degree of consolidation (U') both during and after construction. The solution is divided into two equations:

$$U'(T) = \begin{cases} \dfrac{T}{T_c}\left[1 - \dfrac{1}{T}\sum\limits_{m=0}^{\infty}\dfrac{2}{M^4}\left(1 - e^{-M^2T}\right)\right] \ldots\ldots\ldots T \le T_c \\ 1 - \dfrac{1}{T_C}\sum\limits_{m=0}^{\infty}\dfrac{2}{M^4}\left(e^{M^2T_c} - 1\right)e^{-M^2T} \ldots\ldots T \ge T_c \end{cases} \tag{5}$$

where M and T were formerly defined in Equations 2 and 3, respectively.

Mota (1996) and Hanna et al. (2013) showed that Terzaghi's (1943) empirical method tends to overestimate the average degree of consolidation when compared with Olson's (1977) solution; the higher the value of T_c, the higher the error. The maximum divergence occurs at $T = T_c$. The method overestimates the average degree of consolidation by approximately 10%.

For this reason, Hanna et al. (2013) proposed a new approach to construct the consolidation curve due to a ramp load. During construction, the analytical development agrees with Olson's (1977). After construction, a simple equation is proposed, assuming that the remaining excess pore water pressure at the end of construction T_c is an instantaneous load applied at $T = T_c$.

This technical note reveals that Hanna et al.'s approach (2013) may lead to considerable errors in predicting the average degree of consolidation after the construction period. A corrected solution to the development is then proposed. Besides, a new simple approximate method is suggested, aiming to be more accurate than Terzaghi's (1943).

2. Method based on instantaneous excess pore pressures

2.1 Hanna et al.'s (2013) proposition

Hanna et al. (2013) discretized the ramp load into infinitesimal increments at a rate of λ per unit of time. As shown in Figure 1, Hanna et al. (2013) assume that after an infinitesimal increment dt, the loading is increased by λdt. At the end of construction, the total applied load (q_C) is λt_c. Each load increment results in an infinitesimal increase in pore pressure (Δu_0), which is assumed constant with depth. At the end of construction (t_c), only part of each increment of pore pressure will have been dissipated.

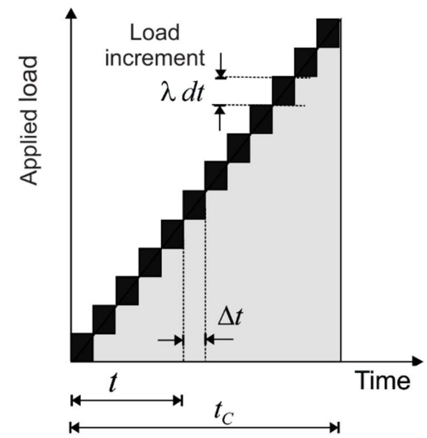

Figure 1. Discretization of the applied load into infinitesimal increments (adapted from Hanna et al., 2013).

Each infinitesimal load increment is applied instantaneously. The dissipation of each excess pore pressure may be expressed by Terzaghi and Fröhlich's (1936) theory. The dissipated excess pore water pressure at the end of loading is $U(t_c - t)\lambda dt$. Considering all intervals, the average degree of consolidation at a time t_c is given by:

$$U'(T_c) = \frac{1}{\lambda t_c}\int_0^{t_c} U(t_c - t)\,\lambda dt = \frac{1}{t_c}\int_0^{t_c} U(t_c - t)\,dt \tag{6}$$

Due to loading linearity (Figure 1), Equation 6 can also be expressed as:

$$U'(T_c) = \frac{1}{t_c}\int_0^{t_c} U(t)\,dt = \frac{1}{T_c}\int_0^{T_c} U(T)\,dT = 1 - \frac{1}{T_c}\sum_{m=0}^{\infty}\frac{2}{M^4}\left(1 - e^{-M^2T_c}\right) \tag{7}$$

By taking T as T_c, Hanna et al. (2013) extended Equation 7 for any time during construction. Thus, the final equation is given by:

$$U'(T \le T_c) = \frac{1}{T_c}\int_0^{T} U(T)\,dT = \frac{T}{T_c}\left[1 - \frac{1}{T}\sum_{m=0}^{\infty}\frac{2}{M^4}\left(1 - e^{-M^2T}\right)\right] \tag{8}$$

which agrees with Olson's (1977) solution during construction (Equation 5).

After the end of construction, Hanna et al. (2013) proposed another methodology. The remaining excess pore pressure $\overline{u_e}$ at the end of construction $T = T_c$ is expressed as a fraction of the final load by:

$$\overline{u_e} = \left[1 - U'(T_c) \right] q_c \tag{9}$$

If $\overline{u_e}$ is interpreted as the excess pore pressure due to an instantaneously applied load, the average degree of consolidation after the end of construction becomes the sum of the corresponding value at the end of loading and the one due to $\overline{u_e}$ dissipation:

$$U'(T \geq T_c) = U'(T_c) + \left[1 - U'(T_c) \right] U(T - T_c) \tag{10}$$

Hanna et al. (2013) applied their method to a practical example of an embankment on a 4 m thick, single-drained clay deposit, with a coefficient of consolidation c_v of 2.0 m^2/year and a coefficient of volume change m_v of 1.2 MPa^{-1}. The construction period was nine months ($T_c = 0.0938$). At the end of construction, the loading achieved 120 kPa. Hanna et al. (2013) calculated the average degrees of consolidation *U' (T)* equal to 12,5% after six months of construction, and equal to 57% after two years.

Figure 2 compares Hanna et al.'s (2013) method with the analytical solutions. After the end of loading, the curve quickly deviates and approaches Terzaghi and Fröhlich's (1936) consolidation curve for instantaneous loading. The results revealed that Hanna et al.'s (2013) proposition is inappropriate after the end of construction.

The error is due to an overestimation of the consolidation rate at the end of construction. Each infinitesimal increase of pore pressure is associated with a different value of the average degree of consolidation. The dissipation of the remaining excess pore pressure at $T = T_c$ is certainly slower than if it was applied instantaneously at that moment.

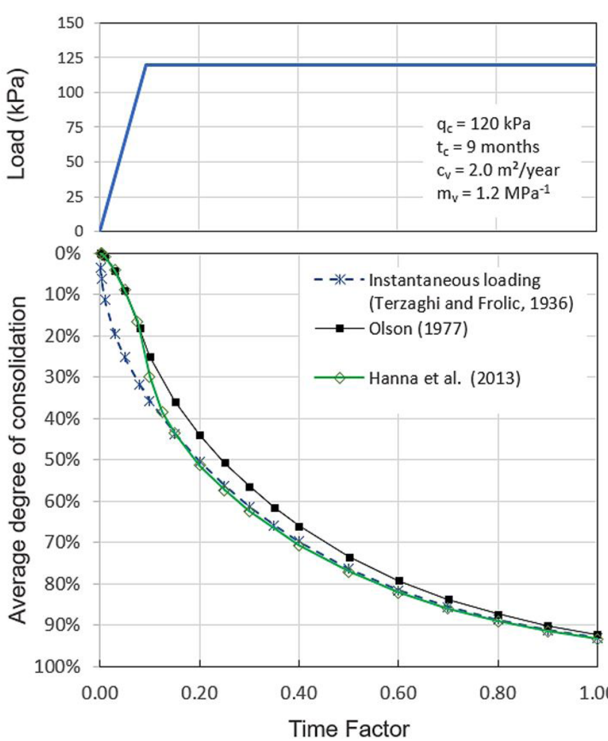

Figure 2. Predictions of the average degree of consolidation due to the construction of an embankment on a 4 m thick, single-drained clay deposit.

2.2 Correcting post-construction consolidation prediction

Hanna et al. (2013)'s procedure may be corrected by the simple application of superposition principle. As shown in Figure 3, the first step consists in a loading extrapolation beyond the end of construction ($t' \geq t_c$) to q'_t. Then, the average degree of consolidation is computed by subtracting from $U'_1(t')$ the corresponding value $U'_2(t')$ due to the excess load; i.e.:

$$U'(t') = U'_1(t') - U'_2(t') \tag{11}$$

The first term $U'_1(t')$ comprises all the real and the virtual infinitesimal load increments. The fraction of the excess pore pressure that is dissipated at any time $t' \geq t_c$ is $U(t' - t)\,\lambda dt$. Thus, for all time intervals, the average degree of consolidation at time t' is given by:

$$U'_1 = \frac{1}{\lambda t_c} \int_0^{t'} U(t' - t)\,\lambda dt = \frac{1}{t_c} \int_0^{t'} U(t)\,dt \tag{12}$$

The second term contains only the virtual load increments. The average degree of consolidation $U'_2(t')$ is determined similarly by shifting the origin of the Cartesian axis. The fraction of excess pore water pressure that is dissipated at time $t' \geq t_c$ is $U(t' - t_c - t)\,\lambda dt$; so:

$$U'_2 = \frac{1}{\lambda t_c} \int_0^{t' - t_c} U(t' - t_c - t)\,\lambda dt = \frac{1}{t_c} \int_0^{t' - t_c} U(t)\,dt \tag{13}$$

Thus, the average degree of consolidation at any time after the end of loading is given by:

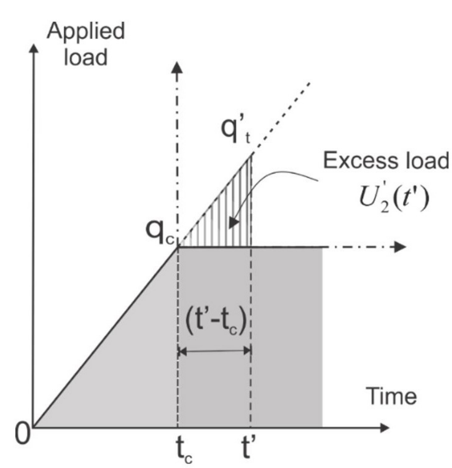

Figure 3. Calculation scheme for $t > t_c$.

$$U'(t') = \frac{1}{t_c}\left[\int_0^{t'} U(t)\,dt - \int_0^{t'-t_c} U(t)\,dt\right] = \frac{1}{t_c}\int_{t'-t_c}^{t'} U(t)\,dt \tag{14}$$

Finally, the average degree of consolidation at any time after the end of construction is expressed by:

$$U'(T \geq T_c) = 1 - \frac{1}{T_c}\sum_{m=0}^{\infty}\frac{2}{M^4}\left[e^{-M^2(T-T_c)} - e^{-M^2 T}\right] \tag{15}$$

This equation is analogous to Olson's (1977) solution (Equation 5) for $t \geq t_c$.

3. A new simple approach to predict the average degree of consolidation

Alternatively, a new simple procedure is proposed to calculate the average degree of consolidation. Its goal is to be as simple and accurate as possible.

3.1 During construction

An approximate solution for the definite integral in Equation 7 can be obtained by numerical integration methods, such as Simpson's rule (Davis and Rabinowitz, 1984). Given three points, Simpson's rule approximates the integrand into a quadratic function.

Applying Simpson's rule on Equation 7, the approximated average degree of consolidation can be expressed by the function values at the lower limit, midpoint, and upper limit:

$$\int_{t_a}^{t_b} U(t)\,dt \approx \frac{T_b - T_a}{6}\left[U(T_a) + 4U\left(\frac{T_b + T_a}{2}\right) + U(T_b)\right] \tag{16}$$

The loading process initiates at $t_a = 0$. Thus, at any Time Factor (T) during construction, the approximate value of the average degree of consolidation is given by:

$$U'(T \leq T_c) = \frac{1}{T_c}\int_0^T U(T)\,dT \approx \frac{T}{T_c}\left[\frac{U(0) + 4U\left(\frac{T}{2}\right) + U(T)}{6}\right] \tag{17}$$

It is worth noting that Equation 17 incorporates an error that increases with the decrease in the rate of loading. If T_c tends to infinity, the average degree of consolidation at the end of construction should be 100%, since consolidation and loading would occur simultaneously. However, Equation 17 gives $U'(T_c) = 5/6(83.3\%)$, since $U(0) = 0$.

To overcome this inherent error, a slight adjustment on the first term in Equation 16 is recommended, as shown in Equation 18. This correction improves the accuracy of Equation 17 and it has no significant influence on predicting the average degree of consolidation for any speed of construction.

$$U'(T \leq T_c) \approx \frac{T}{T_c}\left[\frac{U\left(\frac{T}{24}\right) + 4U\left(\frac{T}{2}\right) + U(T)}{6}\right] \tag{18}$$

It is worth noting that Equation 18 is close to Terzaghi's (1943) graphical method for $T \leq T_c$ (Equation 4), but it provides average degrees of consolidation that are always lower. As shown in Figure 4, the higher the value of T_c, the higher the difference between Terzaghi's (1943) curve and both Olson's (1977) and the method herein proposed.

The error of the approximate propositions relative to Olson's (1977) theoretical solution can be expressed by:

$$Relative\,error = \frac{\Delta U'}{U'_{theoretical}} = \frac{U'_{approximate} - U'_{theoretical}}{U'_{theoretical}} \tag{19}$$

Figure 5 compares the relative error of the two approximate methods. The relative error of Terzaghi's (1943) method reaches magnitudes that cannot be neglected, exceeding 10% for Time Factors T_c higher than 1.0. The proposed approach is less sensitive to the construction duration, with an acceptable error close to 1%.

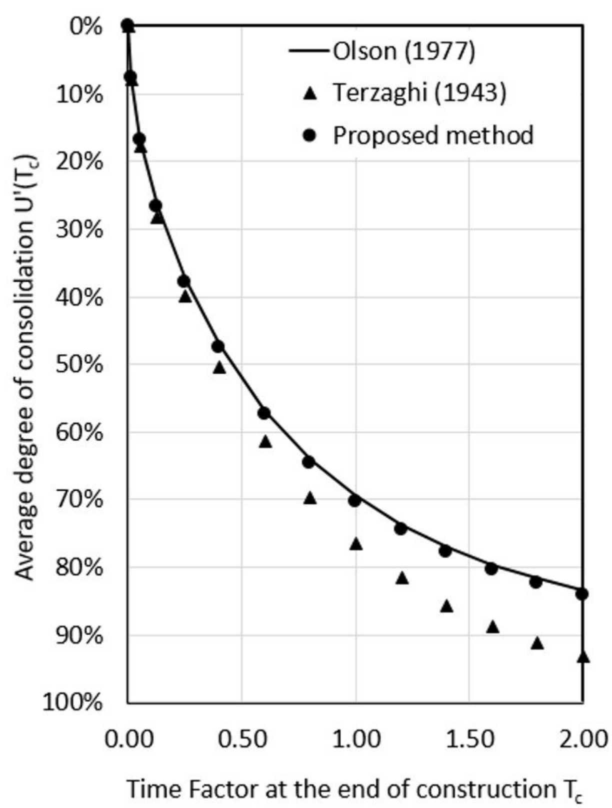

Figure 4. Influence of the construction period on the average degree of consolidation prediction at the end of construction $U'(T_c)$ for ramp loads.

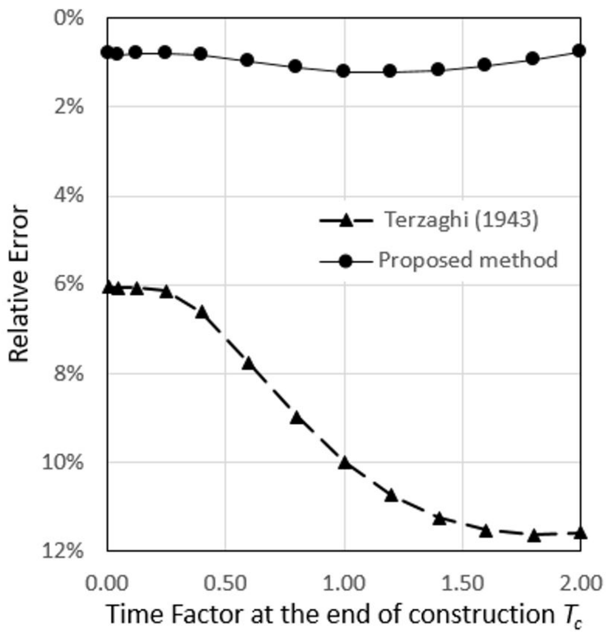

Figure 5. Relative error of the approximate propositions for ramp load.

3.2. After the end of construction

Since the load no longer varies, a procedure similar to Terzaghi's (1943) can be used. The consolidation is approximated considering that the load was instantaneously applied at a Time Factor $T \le T_c$. Thus, the corrected average degree of consolidation can be estimated by determining which Time Fator $T^* \le T_c$ would provide the same average degree of consolidation at the end of construction. In other words, according to Equation 18, $U(T^*)$ is given by:

$$U\left(T^*\right) = U'(T_c) = \left[\frac{U\left(\frac{T_c}{24}\right) + 4U\left(\frac{T_c}{2}\right) + U\left(T_c\right)}{6}\right] \tag{20}$$

This equivalent instantaneous loading was therefore applied at $T = T_c - T^*$. After the end of construction, the instantaneous loading settlement curve is always shifted by $T_c - T^*$. This procedure leads to:

$$U'(T > T_c) \approx U\left(T + T^* - T_c\right) \tag{21}$$

As expected, both Equation 18 and Equation 21 predict the same average degree of consolidation at the end of construction $T = T_c$.

The proposed method was applied to Hanna et al.'s (2013) example (embankment on a 4 m thick, single-drained clay deposit). After six months ($T = 0.0625$), one has:

$$U'(0.0625) = \frac{0.0625}{0.0938}\left[\frac{28.2\% + 4 \cdot 19.9\% + 5.8\%}{6}\right] = 12,6\%$$

After nine months, at the end of construction ($T_c = 0.0938$), one has:

$$U'(0.0938) = \frac{0.0938}{0.0938}\left[\frac{34,5\% + 4 \cdot 24.4\% + 7.1\%}{6}\right] = 23.2\%$$

And after two years ($T = 0.25$), one has:

$$U\left(T^*\right) = 23.2\% \rightarrow T^* = 0.0423$$

$$U'(0.25) = U(0.25 + 0.0423 - 0.0938) = 50.2\%$$

Olson's (1977) exact solution for $T_c = 0.0938$ gives $U'(0.0625) = 12.5\%$, $U'(0.0938) = 23.0\%$ and $U'(0.25) = 50.7\%$.

4. Predicting laboratory test results

The accuracy of the proposed method was also verified in its ability to predict the experimental oedometer test curves.

Sivakugan et al. (2014) carried out laboratory oedometer tests with ramp loading on an artificially mixed kaolinite/sand blend. The ramp loading was performed by filling a bucket, located on the loading arm, with sand scoops over varying periods.

Figure 6 shows the oedometer test data for a 2 hours loading test. The specimen was 18.241 mm thick, and the coefficient of consolidation was $c_v = 0.6$ m^2/year, determined by conventional oedometer tests on the same soil with instantaneous loading. Total stress increase was 215.1 kPa, and settlement at the end of loading was $\rho_c = 0.22$ mm. Experimental normalized settlement was defined as the ratio of vertical displacement ρ to final settlement ρ_c. Estimated normalized settlement, based on the estimations from the distinct propositions shown in previous sections, was calculated as the ratio of estimated average degree of consolidation U' to U'_c at the end of loading ($T_c = 1.6$). It worth noting that the average degree of consolidation U'_c is around 80% for both methods.

There is a reasonable agreement, despite the slight difference between the experimental and numerical results. Olson's (1977) theory and the proposed method only include primary consolidation. If the specimen develops secondary consolidation, the final settlement is higher than the primary compression value. As a result, the experimental normalized settlement becomes lower than the predicted ones. The maximum difference is 5.25%, at $T/T_c = 0.45$.

Mota (1996) performed laboratory tests with ramp loading on 2 cm thick specimens of very soft clay from Barra da Tijuca, Rio de Janeiro, Brazil. Soil characterization revealed natural water content ranging from 132% to 626%. Liquid and plastic limits range from 64% to 488% and 36% to 214%, respectively. The ramp loading test was performed

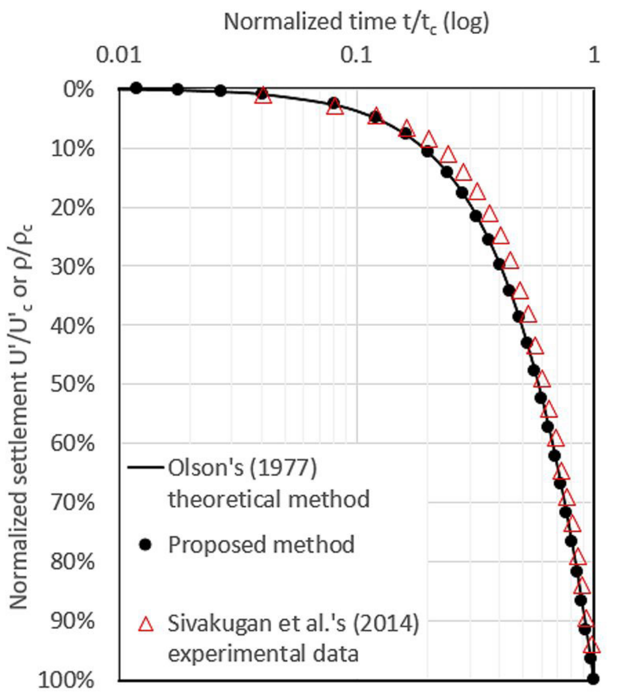

Figure 6. Predicted and measured normalized settlements versus normalized time.

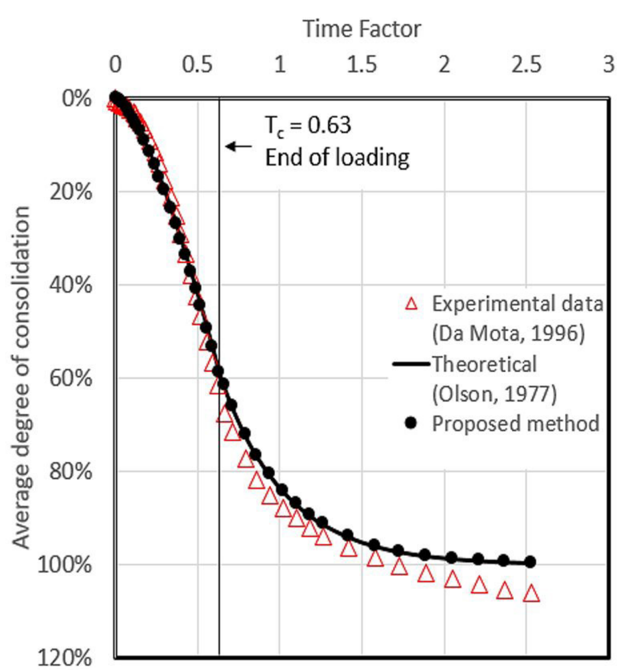

Figure 7. Predicted and measured average degree of consolidation versus Time Factor for ramp load.

by filling a bucket on the loading arm in steps of 1% of the total load at each 1% of the total period of loading.

The coefficient of consolidation was $c_v = 4.3 \cdot 10^{-5}$ cm²/s. Both the coefficient of consolidation and the vertical displacement corresponding to the end of primary (U' = 100%) were determined via Taylor's method in a conventional oedometer test.

Figure 7 compares the experimental data with Olson's (1977) theoretical solution and the proposed method for a 2 hours loading test (T_c = 0.63) and a total stress increase of 100 kPa. The analytical and proposed methods agreed with the experimental results, although small deviations are observed after approximately 60% of primary consolidation. The differences are attributed mainly due to secondary consolidation. At a Time Factor of 2.5, the experimental curve reaches an average degree of consolidation of 106%.

5. Conclusions

This technical note revisited some approximate methods for predicting consolidation settlements due to ramp loading. Terzaghi's method (1943) has shown to be accurate only for small values of T_c values (T_c < 0.2). Hanna et al.'s (2013) approach provides exact results during construction, but it leads to significant errors after the end of construction.

Two procedures have been proposed herein to overcome these issues. The first one improved Hanna et al.'s (2013) approach by combining the applied load discretization with the concept of superposition effects. The correction solved the inaccuracies. The new set of equations was identical to Olson's (1977) solution for any construction time.

Finally, a new approximate method was developed. Simple and easy to apply, it revealed to be much more accurate than Terzaghi's (1943) method when compared to Olson's (1977) solution. Numerical examples have shown that the difference between the proposed method and Olson's (1977) theory is negligible for the whole time range.

The approximate method was also validated by very good reproductions of oedometer tests results in clayey soils subjected to ramp loading. The differences were mainly due to secondary consolidation.

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Declaration of interest

The authors declare no conflict of interest.

Author's contributions

Raphael F. Carneiro: conceptualization, data curation, formal analysis, investigation, methodology, validation, writing – original draft. Denise M. S. Gerscovich: investigation, validation, visualization, writing – review & editing. Bernadete R. Danziger: validation, writing – review & editing.

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List of symbols

H_d maximum length of drainage path;
M count parameter;
T Time factor;
T' Time factor at time t';
T_c Time factor at the end of construction;
T^* a Time Factor such that $U(T^*) = U'(T_c)$;
U average degree of consolidation (instantaneous loading);
U' average degree of consolidation (non-instantaneous loading);
U'_c average degree of consolidation at the end of loading;
c_v coefficient of consolidation;
dt time increment;
m_v coefficient of volume change;
q_c total load;
t time;
t_a lower integral limit;
t_b upper integral limit;
t_c time at the end of construction;
t' any time after the end of construction;
$\overline{u_e}$ remaining excess pore pressure at the end of construction;
λ rate of loading;
ρ settlement;
ρ_c settlement at the end of loading.